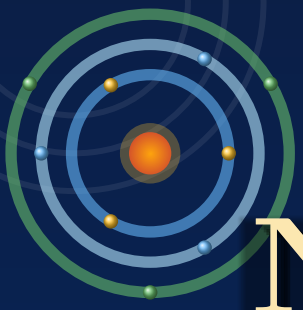

NUCLEAR PHYSICS

Principles and Applications

An Undergraduate Textbook

Author

Dr. Irfan Toqeer

PhD Physics, MSc Computer Science

Assistant Professor
Department of Physics
Riphah International University, Faisalabad

Nuclear Physics

Principles and Applications

Dr. Irfan Toqeer

PhD Physics, MSc Computer Science

Department of Physics

Riphah International University, Faisalabad Campus, Pakistan

"Advancing Knowledge through Physics Education and Research"

Dedicated to

My beloved mother

and my supportive wife

Whose love, prayers, and encouragement make every achievement possible.

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Most importantly, I owe everything to my family — for their patience, prayers, and unwavering faith that guided me throughout this journey.

Preface

Nuclear Physics is written to serve as a core textbook for undergraduate and graduate students of physics and applied sciences. It presents a unified introduction to the fundamental concepts, models, and experimental aspects of nuclear physics. The text emphasizes conceptual clarity, mathematical formulation, and physical interpretation with carefully designed examples.

The chapters follow a natural pedagogical sequence: from the discovery of the nucleus to nuclear models, reactions, fission, fusion, and applications. Each chapter begins with key learning objectives and concludes with a concise summary and a set of problems for self-assessment.

The author hopes that this book will not only serve as a comprehensive course companion but will also inspire further exploration in the fascinating world of nuclear science.

Dr. Irfan Toqeer
Department of Physics
Riphah International University, Faisalabad Campus
irfan.toqeer@riphahfsd.edu.pk

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Conventions and Notation

This book uses SI units throughout unless otherwise stated. The following conventions are adopted:

- Scalars are written in italics (e.g., E, m, r), and vectors in boldface (e.g., $\mathbf{p}$, $\mathbf{r}$).
- Energies are expressed in electron-volts (eV, MeV) unless otherwise specified.
- Natural constants are taken from CODATA 2022 values.
- Equations of special importance are boxed for quick reference.
- References and suggested readings appear at the end of each chapter.

A complete list of symbols and abbreviations is provided at the end of the book. Readers are encouraged to consult it as needed while working through the material.

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Chapter 1

Introduction to Nuclear Physics

1.1 Introduction

Nuclear physics is one of the most fascinating and profound branches of modern physics. It deals with the constituents and structure of atomic nuclei, their properties, interactions, and the fundamental forces that govern their behavior. The discovery of the atomic nucleus in the early 20th century revolutionized our understanding of matter and energy, leading to profound technological advances and deepening our comprehension of the universe itself.

This chapter provides a comprehensive introduction to nuclear physics, beginning with its historical development and the key experiments that led to the discovery of the nucleus. We will explore the fundamental particles that constitute nuclear matter, understand the units and dimensions specific to nuclear physics, and gain an overview of the nuclear forces that bind nuclei together.

The study of nuclear physics is not merely of academic interest. It has profound practical applications ranging from nuclear energy generation and medical diagnostics to our understanding of stellar processes and the origin of elements in the universe. A solid foundation in nuclear physics is essential for students pursuing careers in physics, engineering, medicine, and related fields.

1.2 Historical Background and Discovery of the Nucleus

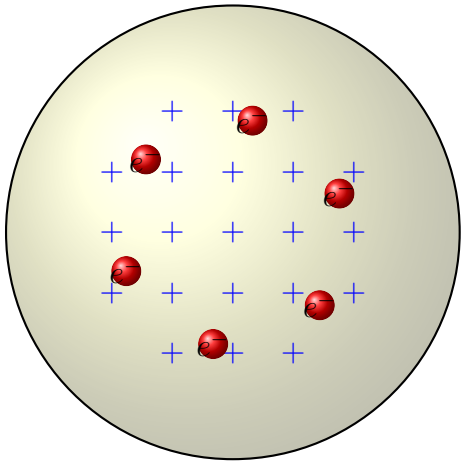
1.2.1 Early Atomic Models

The journey toward understanding the atomic nucleus began with attempts to understand the structure of atoms themselves. In the late 19th century, atoms were thought to be indivisible fundamental particles, as suggested by John Dalton's atomic theory (1808). However, the discovery of the electron by J.J. Thomson in 1897 challenged this view and demonstrated that atoms have internal structure.

Thomson's Plum Pudding Model

Following his discovery of the electron, J.J. Thomson proposed the "plum pudding" model of the atom in 1904. In this model:

- The atom was conceived as a sphere of positive charge
- Electrons were embedded within this positive sphere, like plums in a pudding
- The positive and negative charges balanced to make the atom electrically neutral
- The atom had a radius of approximately 10^{-10} meters



Thomson's Plum Pudding Model

Figure 1.1: Thomson's plum pudding model showing electrons embedded in a sphere of positive charge

While this model successfully explained some electrical properties of atoms, it would soon be challenged by experimental evidence.

1.2.2 The Rutherford Scattering Experiment

The most pivotal experiment in the discovery of the atomic nucleus was conducted by Ernest Rutherford and his colleagues Hans Geiger and Ernest Marsden between 1909 and 1911 at the University of Manchester.

Experimental Setup

The experiment involved:

1. A radioactive source (radium or polonium) that emitted alpha particles (α-particles, which are helium nuclei with charge $+2e$)
2. A thin gold foil (approximately 10^{-6} m or 1000 atoms thick)
3. A zinc sulfide screen or detector to observe scattered alpha particles
4. The apparatus was enclosed in a vacuum chamber to prevent air scattering

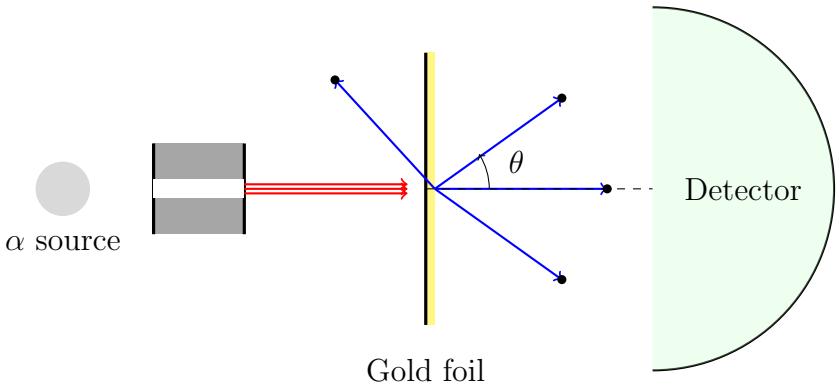


Figure 1.2: Schematic diagram of the Rutherford scattering experiment

Expected vs. Observed Results

According to Thomson's plum pudding model, the alpha particles should pass through the gold foil with minimal deflection, as the positive charge was thought to be spread uniformly throughout the atom. The maximum expected deflection angle was only a few degrees.

Key Observations

1. Most alpha particles (about 99.99%) passed straight through the foil with little or no deflection
2. A small fraction (about 1 in 8000) were deflected at angles greater than 90
3. Some particles were even scattered backward (deflection angle > 150)

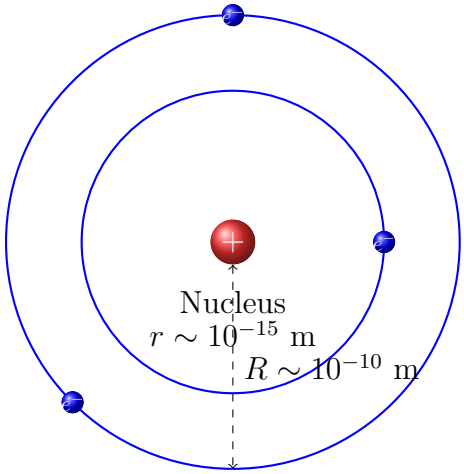
Rutherford famously described his surprise at these results:

> *"It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."*

The Nuclear Model

To explain these observations, Rutherford proposed a new atomic model in 1911:

- The atom consists of a tiny, dense, positively charged nucleus
- The nucleus contains most of the atom's mass
- Electrons orbit the nucleus at relatively large distances
- Most of the atom's volume is empty space
- The nuclear radius is approximately 10^{-15} to 10^{-14} meters
- The atomic radius is approximately 10^{-10} meters



Rutherford's Nuclear Model

Figure 1.3: Rutherford's nuclear model of the atom

1.2.3 Mathematical Analysis of Rutherford Scattering

Rutherford developed a mathematical theory to describe the scattering of alpha particles by atomic nuclei. This theory is based on classical mechanics and the Coulomb interaction between charged particles.

Scattering Geometry

Consider an alpha particle with charge $Z_\alpha e = 2e$, mass m_α, and initial kinetic energy K approaching a nucleus with charge Ze (where Z is the atomic number).

The Coulomb potential energy between the alpha particle and nucleus is:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Z_\alpha Z e^2}{r} = \frac{k Z_\alpha Z e^2}{r} \tag{1.1}$$

where $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9$ N·m^2/C^2 is Coulomb's constant.

The relationship between the impact parameter b (the perpendicular distance from the nucleus to the initial trajectory of the alpha particle) and the scattering angle θ is derived from conservation of energy and angular momentum:

$$b = \frac{d}{2}\cot\left(\frac{\theta}{2}\right) \tag{1.2}$$

where d is the distance of closest approach for a head-on collision ($b = 0$):

$$d = \frac{k Z_\alpha Z e^2}{K} \tag{1.3}$$

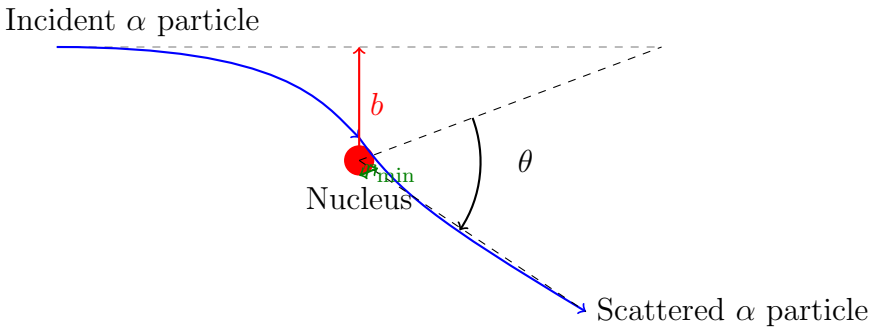


Figure 1.4: Geometry of Rutherford scattering showing impact parameter b and scattering angle θ

The Rutherford Scattering Formula

The differential scattering cross-section describes the probability that an alpha particle will be scattered into a solid angle $d\Omega$ at angle θ. Rutherford derived:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_\alpha Z e^2}{8\pi\epsilon_0 K}\right)^2 \frac{1}{\sin^4(\theta/2)} \tag{1.4}$$

This can also be written as:

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{d}{4}\right)^2 \frac{1}{\sin^4(\theta/2)}} \tag{1.5}$$

where d is the distance of closest approach defined earlier.

The number of particles $N(\theta)$ scattered at angle θ into a detector subtending solid angle $\Delta\Omega$ is:

$$N(\theta) = N_i n t \frac{d\sigma}{d\Omega}\Delta\Omega \tag{1.6}$$

where:

- N_i is the number of incident alpha particles
- n is the number density of nuclei in the target (nuclei per unit volume)
- t is the thickness of the target foil

Key Predictions of Rutherford Formula

The scattering formula predicts that the number of scattered particles:

1. Is proportional to $\frac{1}{\sin^4(\theta/2)}$ (very strong angular dependence)
2. Is proportional to Z^2 (square of nuclear charge)
3. Is proportional to $\frac{1}{K^2}$ (inversely proportional to square of kinetic energy)
4. Is proportional to t (target thickness)

Example 1.2.1. Estimating Nuclear Size

Consider a 5.0 MeV alpha particle ($Z_\alpha = 2$) incident on a gold nucleus ($Z = 79$). Calculate the distance of closest approach.

Solution:

Given:

- $K = 5.0$ MeV $= 5.0 \times 1.602 \times 10^{-13}$ J $= 8.01 \times 10^{-13}$ J
- $Z_\alpha = 2$, $Z = 79$
- $e = 1.602 \times 10^{-19}$ C
- $k = 8.99 \times 10^9$ N·m^2/C^2

Using equation (1.3):

$$\begin{aligned} d &= \frac{kZ_\alpha Ze^2}{K} \\ &= \frac{(8.99 \times 10^9)(2)(79)(1.602 \times 10^{-19})^2}{8.01 \times 10^{-13}} \\ &= \frac{3.65 \times 10^{-26}}{8.01 \times 10^{-13}} \\ &= 4.56 \times 10^{-14} \text{ m} \\ &= 45.6 \text{ fm} \end{aligned}$$

This distance is much larger than the actual gold nucleus radius (≈ 7 fm), indicating that the alpha particle does not actually touch the nucleus in this case. The scattering is purely electromagnetic.

1.2.4 Discovery of the Neutron

While Rutherford's nuclear model successfully explained atomic structure, several puzzles remained:

- Atomic masses were approximately twice what would be expected from the nuclear charge alone
- For example, helium has $Z = 2$ but atomic mass number $A = 4$

Initially, it was proposed that nuclei contained protons and electrons. However, this model had serious problems:

1. The Heisenberg uncertainty principle suggested that confining electrons to nuclear dimensions would require impossibly high energies
2. Nuclear spin measurements were inconsistent with the proton-electron model
3. The statistics of nuclei (Bose-Einstein vs. Fermi-Dirac) did not match predictions

Chadwick's Experiment (1932)

The neutron was discovered by James Chadwick in 1932 through the following observations:

1. Irène Curie and Frédéric Joliot found that when beryllium was bombarded with alpha particles, a penetrating radiation was produced
2. This radiation could knock protons out of paraffin wax (which contains hydrogen)
3. Initially thought to be gamma rays, but the energetics didn't work out

The nuclear reaction involved was:

$$^{9}\text{Be} + ^{4}\text{He} \rightarrow ^{12}\text{C} + ^{1}\text{n} \tag{1.7}$$

Chadwick demonstrated that this radiation consisted of neutral particles with mass approximately equal to the proton mass. He named these particles *neutrons*.

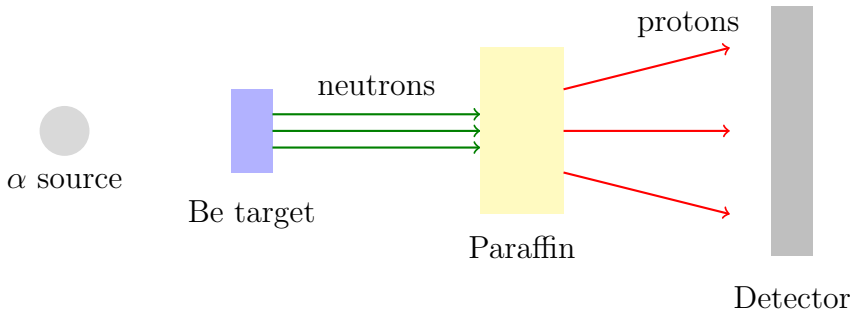


Figure 1.5: Schematic of Chadwick's neutron discovery experiment

Properties of the Neutron

- Symbol: n or ${}^{1}_{0}\text{n}$
- Charge: 0 (electrically neutral)
- Mass: $m_n = 1.675 \times 10^{-27}$ kg $= 939.565$ MeV/c^2
- Spin: 1/2 (fermion)
- Slightly more massive than proton: $m_n - m_p \approx 1.29$ MeV/c^2
- Free neutrons are unstable: $\tau_{1/2} \approx 10.2$ minutes

The discovery of the neutron completed our basic understanding of nuclear structure: nuclei consist of protons and neutrons, collectively called *nucleons*.

1.3 Fundamental Particles: Proton, Neutron, Electron, Neutrino

In this section, we examine in detail the fundamental particles that are essential to nuclear physics and nuclear processes.

1.3.1 The Proton

The proton is a positively charged subatomic particle and one of the primary constituents of atomic nuclei.

Properties of the Proton

Property	Value
Symbol	p or ${}^{1}_{1}\mathrm{H}$ or ${}^{1}_{1}\mathrm{p}$
Charge	$+e = +1.602 \times 10^{-19}$ C
Mass	$m_p = 1.673 \times 10^{-27}$ kg
	$m_p = 1.007276$ u
	$m_p = 938.272$ MeV/c^2
Spin	1/2 (fermion)
Magnetic moment	$\mu_p = 2.793\mu_N$
Stability	Stable (no observed decay)

Table 1.1: Properties of the proton

The proton is actually a composite particle made of three quarks (two up quarks and one down quark: *uud*), but for most nuclear physics applications, we treat it as a fundamental point-like particle.

1.3.2 The Neutron

As discussed earlier, the neutron is the neutral partner of the proton in nuclear matter.

Properties of the Neutron

Property	Value
Symbol	n or ${}^{1}_{0}\mathrm{n}$
Charge	0
Mass	$m_n = 1.675 \times 10^{-27}$ kg
	$m_n = 1.008665$ u
	$m_n = 939.565$ MeV/c^2
Spin	1/2 (fermion)
Magnetic moment	$\mu_n = -1.913\mu_N$
Half-life (free)	$t_{1/2} = 10.2$ minutes

Table 1.2: Properties of the neutron

Neutron Decay

Free neutrons are unstable and undergo beta decay:

$$n \to p + e^- + \bar{\nu}_e \tag{1.8}$$

This process is called *beta-minus decay*. The decay has a half-life of approximately 10.2 minutes for free neutrons. However, neutrons bound in stable nuclei do not decay

because the decay would violate energy conservation (the final state would have higher mass-energy).

The energy released in neutron decay can be calculated from the mass difference:

$$Q = (m_n - m_p - m_e)c^2 \tag{1.9}$$
$$= (939.565 - 938.272 - 0.511)\ \text{MeV} \tag{1.10}$$
$$= 0.782\ \text{MeV} \tag{1.11}$$

1.3.3 The Electron

The electron plays a crucial role in atomic physics and is also produced in nuclear beta decay processes.

Properties of the Electron

Property	**Value**
Symbol	e^- or β^-
Charge	$-e = -1.602 \times 10^{-19}$ C
Mass	$m_e = 9.109 \times 10^{-31}$ kg
	$m_e = 0.000549$ u
	$m_e = 0.511$ MeV/c^2
Spin	1/2 (fermion)
Lepton number	+1
Stability	Stable

Table 1.3: Properties of the electron

The electron is a truly fundamental particle (not composed of quarks) and belongs to the class of particles called *leptons*.

1.3.4 The Neutrino

The neutrino is one of the most elusive particles in physics, interacting only through the weak nuclear force and gravity.

The Neutrino Hypothesis

In the 1930s, careful studies of beta decay revealed a puzzle:

- In the decay $n \to p + e^-$, energy and momentum did not appear to be conserved
- Electrons were emitted with a continuous spectrum of energies, not discrete values
- Angular momentum (spin) did not balance

In 1930, Wolfgang Pauli proposed a desperate solution: an additional, undetected particle was being emitted in beta decay. This particle would have to have the following properties:

- Zero or very small mass
- Zero electric charge
- Spin 1/2
- Extremely weak interaction with matter

Enrico Fermi named this hypothetical particle the *neutrino* (Italian for "little neutral one") and incorporated it into his theory of beta decay in 1934.

Detection of the Neutrino

The neutrino was finally directly detected by Clyde Cowan and Frederick Reines in 1956, more than 25 years after Pauli's hypothesis. They used a nuclear reactor as an intense source of antineutrinos and detected the inverse beta decay reaction:

$$\bar{\nu}_e + p \rightarrow n + e^+ \tag{1.12}$$

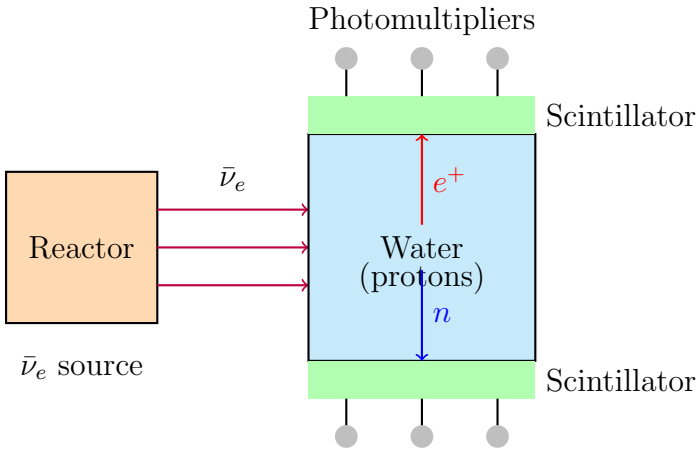


Figure 1.6: Schematic of Cowan-Reines neutrino detection experiment

Properties of Neutrinos

There are three known types (flavors) of neutrinos, each associated with a charged lepton:

- Electron neutrino: ν_e
- Muon neutrino: ν_μ
- Tau neutrino: ν_τ

Each neutrino has a corresponding antiparticle (antineutrino): $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$.

Property	Value
Symbol	ν (ν_e, ν_μ, ν_τ)
Charge	0
Mass	Very small, < 1 eV/c^2 (non-zero!)
Spin	1/2 (fermion)
Lepton number	+1 (for neutrino), -1 (for antineutrino)
Interactions	Weak force only (+ gravity)

Table 1.4: Properties of neutrinos

Note: Neutrino Oscillations

Neutrinos can change from one flavor to another as they propagate through space, a phenomenon called *neutrino oscillations*. This was discovered in the late 1990s and implies that neutrinos must have non-zero (though very small) masses. The 2015 Nobel Prize in Physics was awarded for this discovery.

1.3.5 Comparison of Fundamental Particles

Particle	Symbol	Charge	Mass (MeV/c^2)	Spin
Proton	p	+1	938.272	1/2
Neutron	n	0	939.565	1/2
Electron	e^-	-1	0.511	1/2
Neutrino	ν	0	< 0.000001	1/2

Table 1.5: Comparison of fundamental particles in nuclear physics

1.4 Units and Dimensions in Nuclear Physics

Nuclear physics deals with phenomena at extremely small length scales and high energies. Conventional SI units (meters, kilograms, seconds) are often inconvenient at these scales. Special units have been developed for nuclear physics.

1.4.1 Length: The Fermi (Femtometer)

The basic unit of length in nuclear physics is the **fermi** (fm), also called the **femtometer**:

$$1 \text{ fm} = 10^{-15} \text{ m} \tag{1.13}$$

This unit is named after Enrico Fermi. It is approximately the size of a nucleon (proton or neutron).

Typical length scales in nuclear physics:

- Nucleon radius: ~ 0.8 fm
- Light nucleus (e.g., ^{12}C) radius: ~ 3 fm
- Heavy nucleus (e.g., ^{238}U) radius: ~ 7 fm

Chapter 2

Basic Nuclear Properties

2.1 Introduction

Having explored the historical development and fundamental particles of nuclear physics in Chapter 1, we now turn our attention to the intrinsic properties of atomic nuclei themselves. The atomic nucleus, despite occupying only a tiny fraction of atomic volume, contains more than 99.9% of the atom's mass and determines the chemical identity of elements.

Understanding nuclear properties is fundamental to nuclear physics. These properties include the composition, size, shape, mass, binding energy, and stability of nuclei. The study of these characteristics reveals the nature of nuclear forces and provides insights into nuclear reactions, radioactive decay, and energy generation in stars.

In this chapter, we will develop a quantitative understanding of:

- The composition and structure of nuclei
- Methods for measuring nuclear size and the empirical radius formula
- Nuclear mass, binding energy, and the origin of nuclear stability
- The mass defect and its relationship to binding energy
- The semi-empirical mass formula (SEMF) and its physical basis
- The binding energy curve and the valley of stability

These concepts form the foundation for understanding nuclear reactions, radioactive decay, fission, and fusion—topics that will be explored in later chapters.

2.2 Nuclear Composition, Size, and Density

2.2.1 Nuclear Composition and Notation

As established in Chapter 1, atomic nuclei are composed of two types of particles, collectively called **nucleons**:

- **Protons (p)**: Positively charged particles with charge $+e$
- **Neutrons (n)**: Electrically neutral particles

Standard Nuclear Notation

A nucleus is uniquely specified by two numbers:

- **Atomic number (Z)**: The number of protons
- **Mass number (A)**: The total number of nucleons (protons + neutrons)

The number of neutrons is given by:

$$N = A - Z \tag{2.1}$$

The standard notation for a nucleus is:

$$^{A}_{Z}X_{N} \tag{2.2}$$

where X is the chemical symbol. Often, the neutron number N is omitted, giving:

$$^{A}_{Z}X \tag{2.3}$$

Since the chemical symbol already specifies Z, a simplified notation is commonly used:

$$^{A}X \quad \text{or} \quad X\text{-}A \tag{2.4}$$

Example 2.2.1. Nuclear Notation Examples

- $^{12}_{6}C_{6}$ or ^{12}C or C-12: Carbon-12 with 6 protons and 6 neutrons
- $^{238}_{92}U_{146}$ or ^{238}U or U-238: Uranium-238 with 92 protons and 146 neutrons
- $^{56}_{26}Fe_{30}$ or ^{56}Fe or Fe-56: Iron-56 with 26 protons and 30 neutrons

Isotopes, Isotones, and Isobars

Nuclei can be classified based on their values of Z, N, and A:

Definition 2.2.1. Isotopes are nuclei with the same atomic number Z but different mass numbers A (same element, different neutron numbers).

Examples: ^{12}C, ^{13}C, ^{14}C (all have $Z = 6$ but $N = 6, 7, 8$ respectively)

Definition 2.2.2. Isotones are nuclei with the same neutron number N but different atomic numbers Z.

Examples: ^{13}C and ^{14}N (both have $N = 7$)

Definition 2.2.3. Isobars are nuclei with the same mass number A but different atomic numbers Z.

Examples: ^{14}C and ^{14}N (both have $A = 14$)

Definition 2.2.4. Isomers are nuclei with the same Z, N, and A but different energy states (excited states with measurable lifetimes).

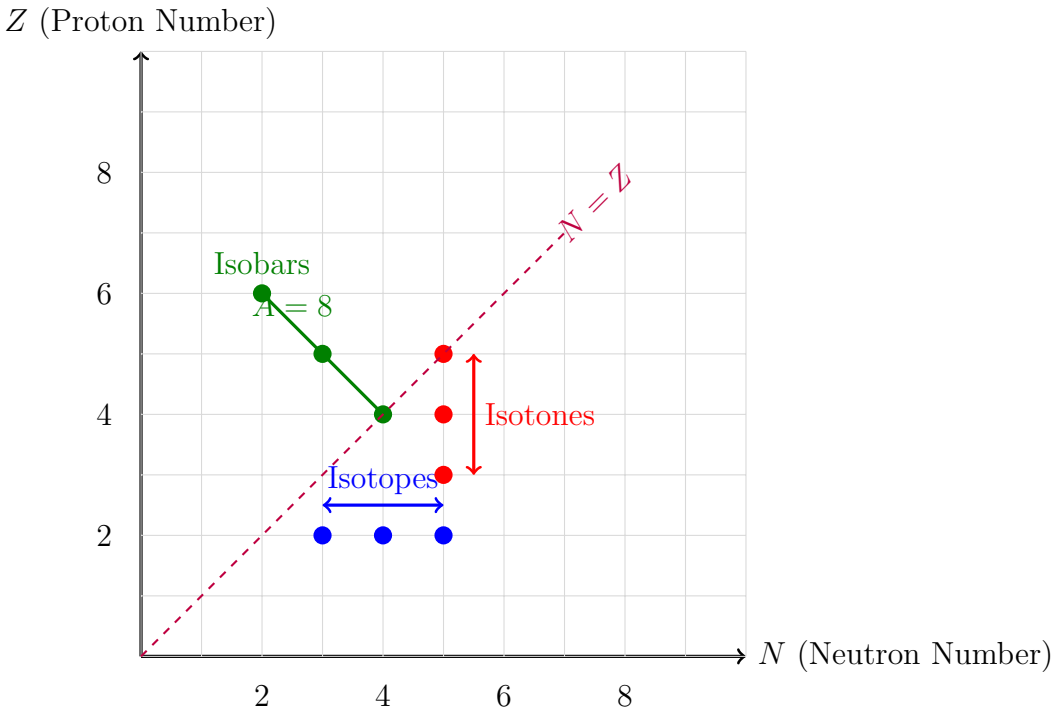


Figure 2.1: Nuclear chart showing isotopes, isotones, and isobars

2.2.2 Nuclear Size

The size of the nucleus is one of its most fundamental properties. Unlike atoms, which don't have well-defined boundaries, nuclei have relatively sharp surfaces that can be characterized by a radius.

Determination of Nuclear Radius

Several experimental methods have been used to measure nuclear radii:

1. **Rutherford Scattering**: Measures the distance of closest approach in Coulomb scattering (gives upper limit)
2. **Electron Scattering**: High-energy electrons probe nuclear charge distribution with high precision
3. **X-ray Spectroscopy**: Measures the energy of transitions in muonic atoms
4. **Neutron Scattering**: Probes the distribution of nuclear matter
5. **Mirror Nuclei**: Analyzes Coulomb energy differences between mirror nuclei

The Empirical Radius Formula

Experiments show that nuclear matter has approximately constant density. This leads to the important result that nuclear volume is proportional to mass number:

$$V \propto A \tag{2.5}$$

Since $V = \frac{4}{3}\pi R^3$, we have:

$$\frac{4}{3}\pi R^3 \propto A \tag{2.6}$$

Therefore:

$$R^3 \propto A \quad \Rightarrow \quad R \propto A^{1/3} \tag{2.7}$$

This gives the empirical nuclear radius formula:

$$\boxed{R = R_0 A^{1/3}} \tag{2.8}$$

where R_0 is a constant with the value:

$$R_0 = 1.2 \text{ fm} = 1.2 \times 10^{-15} \text{ m} \tag{2.9}$$

More precise measurements give R_0 values between 1.1 and 1.3 fm, depending on the measurement method and how the radius is defined.

Key Points about Nuclear Radius

1. Nuclear radius increases with mass number as $A^{1/3}$
2. Doubling the mass number increases radius by only $2^{1/3} \approx 1.26$
3. Heavy nuclei are not much larger than light nuclei
4. The $A^{1/3}$ dependence implies constant nuclear density

Example 2.2.2. Calculating Nuclear Radii

Calculate the radii of the following nuclei:

(a) ^4He (helium-4)

(b) ^{56}Fe (iron-56)

(c) ^{208}Pb (lead-208)

(d) ^{238}U (uranium-238)

Compare the radius of uranium to that of helium.

Solution:

Using $R = R_0 A^{1/3}$ with $R_0 = 1.2$ fm:

(a) For ^4He: $A = 4$

$$R = 1.2 \times 4^{1/3} = 1.2 \times 1.587 = 1.90 \text{ fm}$$

(b) For ^{56}Fe: $A = 56$

$$R = 1.2 \times 56^{1/3} = 1.2 \times 3.826 = 4.59 \text{ fm}$$

(c) For ^{208}Pb: $A = 208$

$$R = 1.2 \times 208^{1/3} = 1.2 \times 5.926 = 7.11 \text{ fm}$$

(d) For ^{238}U: $A = 238$

$$R = 1.2 \times 238^{1/3} = 1.2 \times 6.199 = 7.44 \text{ fm}$$

Ratio of uranium to helium radius:

$$\frac{R_U}{R_{He}} = \frac{7.44}{1.90} = 3.92 \approx 4$$

Even though uranium has 59.5 times more nucleons than helium, its radius is only about 4 times larger!

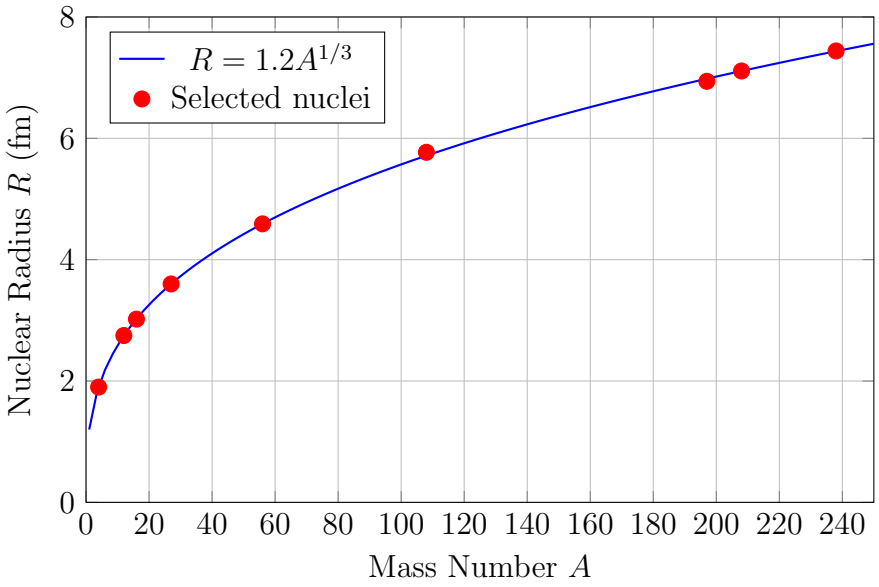


Figure 2.2: Nuclear radius as a function of mass number

2.2.3 Nuclear Density

The fact that $R \propto A^{1/3}$ implies that nuclear matter has approximately constant density, independent of the size of the nucleus.

Calculation of Nuclear Density

The volume of a nucleus is:

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi (R_0 A^{1/3})^3 = \frac{4}{3}\pi R_0^3 A \tag{2.10}$$

The mass of a nucleus with A nucleons is approximately:

$$M \approx A \cdot m_u \tag{2.11}$$

where $m_u \approx 1.66 \times 10^{-27}$ kg is the atomic mass unit (approximately the mass of one nucleon).

The nuclear density is:

$$\rho = \frac{M}{V} = \frac{A \cdot m_u}{\frac{4}{3}\pi R_0^3 A} = \frac{3m_u}{4\pi R_0^3} \tag{2.12}$$

Notice that the mass number A cancels out, confirming that nuclear density is constant!

Example 2.2.3. Calculating Nuclear Density

Calculate the density of nuclear matter.

Solution:

Using:

- $R_0 = 1.2 \times 10^{-15}$ m
- $m_u = 1.66 \times 10^{-27}$ kg

$$\begin{aligned}\rho &= \frac{3m_u}{4\pi R_0^3} \\ &= \frac{3 \times 1.66 \times 10^{-27}}{4\pi \times (1.2 \times 10^{-15})^3} \\ &= \frac{4.98 \times 10^{-27}}{4\pi \times 1.728 \times 10^{-45}} \\ &= \frac{4.98 \times 10^{-27}}{2.17 \times 10^{-44}} \\ &= 2.3 \times 10^{17} \text{ kg/m}^3\end{aligned}$$

This is an extraordinarily large density—about 10^{14} times the density of water!

Nuclear Density

Key facts about nuclear density:

- Nuclear density $\rho \approx 2.3 \times 10^{17}$ kg/m^3
- Nuclear density is approximately constant for all nuclei
- One cubic centimeter of nuclear matter would have a mass of 2.3×10^{14} kg (230 million tonnes!)
- This constancy suggests that nucleons are closely packed and interact only with nearest neighbors (saturation property)

Example 2.2.4. Nuclear Matter in Extreme Conditions

If the Earth (mass $\approx 6 \times 10^{24}$ kg) were compressed to nuclear density, what would be its radius?

Solution:

The volume at nuclear density would be:

$$V = \frac{M}{\rho} = \frac{6 \times 10^{24}}{2.3 \times 10^{17}} = 2.6 \times 10^7 \text{ m}^3$$

The radius would be:

$$\begin{aligned}R &= \left(\frac{3V}{4\pi}\right)^{1/3} = \left(\frac{3 \times 2.6 \times 10^7}{4\pi}\right)^{1/3} \\ &= (6.2 \times 10^6)^{1/3} = 184 \text{ m}\end{aligned}$$

The entire Earth compressed to nuclear density would fit in a sphere of radius only about 180 meters!

Note: Such extreme densities are found in neutron stars.

2.2.4 Nuclear Charge Distribution

The charge distribution in nuclei is not uniform but has been measured precisely using electron scattering experiments. The nuclear charge density can be approximated by various models:

Uniform Sphere Model

The simplest model assumes constant charge density:

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \tag{2.13}$$

Woods-Saxon Distribution

A more realistic model uses the Woods-Saxon (Fermi) distribution:

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/a}} \tag{2.14}$$

where:

- ρ_0 is the central density
- R is the half-density radius
- $a \approx 0.54$ fm is the surface thickness parameter

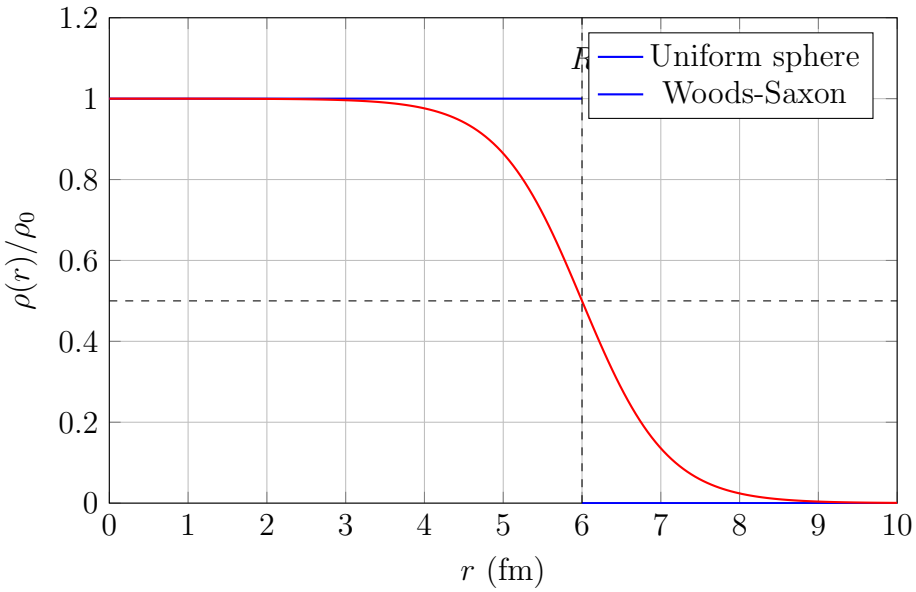


Figure 2.3: Comparison of uniform sphere and Woods-Saxon charge distributions

2.3 Nuclear Mass and Binding Energy

2.3.1 Nuclear Mass

The mass of a nucleus is one of its most fundamental and precisely measurable properties. Nuclear masses are measured with extraordinary precision using mass spectrometry.

Mass Units

As discussed in Chapter 1, nuclear masses are expressed in atomic mass units (u):

$$1 \text{ u} = \frac{1}{12} m(^{12}C) = 1.66054 \times 10^{-27} \text{ kg} = 931.494 \text{ MeV}/c^2 \tag{2.15}$$

Atomic vs. Nuclear Mass

It's important to distinguish between:

- **Nuclear mass** M_{nuc}: Mass of the bare nucleus (protons + neutrons only)
- **Atomic mass** M_{atom}: Mass of the neutral atom (nucleus + electrons)

The relationship is:

$$M_{atom} = M_{nuc} + Zm_e - B_e \tag{2.16}$$

where B_e is the total electronic binding energy (typically a few keV, negligible compared to nuclear masses).

For most nuclear physics calculations, we use atomic masses from tables and account for electron masses appropriately.

2.3.2 Mass Defect

One of the most important discoveries in nuclear physics is that the mass of a nucleus is *less* than the sum of the masses of its constituent nucleons.

Definition 2.3.1. The **mass defect** (ΔM) is the difference between the sum of the masses of separated nucleons and the actual mass of the nucleus:

$$\Delta M = [Zm_p + Nm_n] - M_{nuc} \tag{2.17}$$

Alternatively, using atomic masses:

$$\Delta M = [Zm_H + Nm_n] - M_{atom} \tag{2.18}$$

where m_H is the mass of a hydrogen atom.

Physical Interpretation of Mass Defect

The mass defect represents the mass equivalent of the binding energy that holds the nucleus together. When nucleons combine to form a nucleus:

- Energy is released (binding energy)
- This energy comes from mass via $E = mc^2$
- The resulting nucleus has less mass than its constituents
- The "missing" mass has been converted to binding energy

2.3.3 Nuclear Binding Energy

Definition 2.3.2. The **binding energy** (BE) is the energy required to completely disassemble a nucleus into its constituent protons and neutrons at rest and infinitely separated.

By Einstein's mass-energy relation:

$$\boxed{BE = \Delta M c^2} \tag{2.19}$$

More explicitly:

$$BE = [Zm_p + Nm_n - M_{nuc}]c^2 \tag{2.20}$$

Or using atomic masses:

$$BE = [Zm_H + Nm_n - M_{atom}]c^2 \tag{2.21}$$

Example 2.3.1. Calculating Binding Energy of Deuteron

Calculate the binding energy of the deuteron (^2H), the simplest nucleus after the proton.

Given:

- Mass of ^2H atom: $M(^2H) = 2.014102$ u
- Mass of hydrogen atom: $m_H = 1.007825$ u
- Mass of neutron: $m_n = 1.008665$ u
- Conversion: 1 u = 931.494 MeV/c^2

Solution:

The deuteron has $Z = 1$ proton and $N = 1$ neutron.

Mass defect:

$$\begin{aligned}\Delta M &= [Zm_H + Nm_n] - M(^2H) \\ &= [1 \times 1.007825 + 1 \times 1.008665] - 2.014102 \\ &= 2.016490 - 2.014102 \\ &= 0.002388 \text{ u}\end{aligned}$$

Binding energy:

$$\begin{aligned}BE = \Delta M c^2 &= 0.002388 \text{ u} \times 931.494 \text{ MeV/u} \\ &= 2.224 \text{ MeV}\end{aligned}$$

The binding energy of the deuteron is 2.224 MeV. This is the energy required to break the deuteron into a free proton and neutron.

Example 2.3.2. Binding Energy of ^{56}Fe

Calculate the binding energy and binding energy per nucleon for ^{56}Fe.

Given:

- $M(^{56}Fe) = 55.934937$ u
- $Z = 26$, $N = 30$, $A = 56$

Solution:
Mass defect:

$$\begin{aligned}\Delta M &= [26 \times 1.007825 + 30 \times 1.008665] - 55.934937 \\ &= [26.20345 + 30.25995] - 55.934937 \\ &= 56.46340 - 55.934937 \\ &= 0.528463 \text{ u}\end{aligned}$$

Total binding energy:

$$BE = 0.528463 \times 931.494 = 492.26 \text{ MeV}$$

Binding energy per nucleon:

$$\frac{BE}{A} = \frac{492.26}{56} = 8.79 \text{ MeV/nucleon}$$

Iron-56 has one of the highest binding energies per nucleon, making it one of the most stable nuclei.

2.3.4 Binding Energy per Nucleon

The **binding energy per nucleon** is defined as:

$$\frac{BE}{A} = \frac{\text{Total binding energy}}{\text{Number of nucleons}} \tag{2.22}$$

This quantity is a measure of the average energy needed to remove a nucleon from the nucleus, and thus a measure of nuclear stability.

Significance of BE/A

- Higher BE/A means greater stability
- Nuclei with maximum BE/A are the most tightly bound
- BE/A varies from about 1 MeV for deuteron to about 8.8 MeV for iron
- The variation of BE/A with mass number explains fusion and fission

2.4 The Binding Energy Curve

2.4.1 Variation of Binding Energy with Mass Number

The binding energy per nucleon varies systematically with mass number A. This variation is one of the most important curves in nuclear physics.

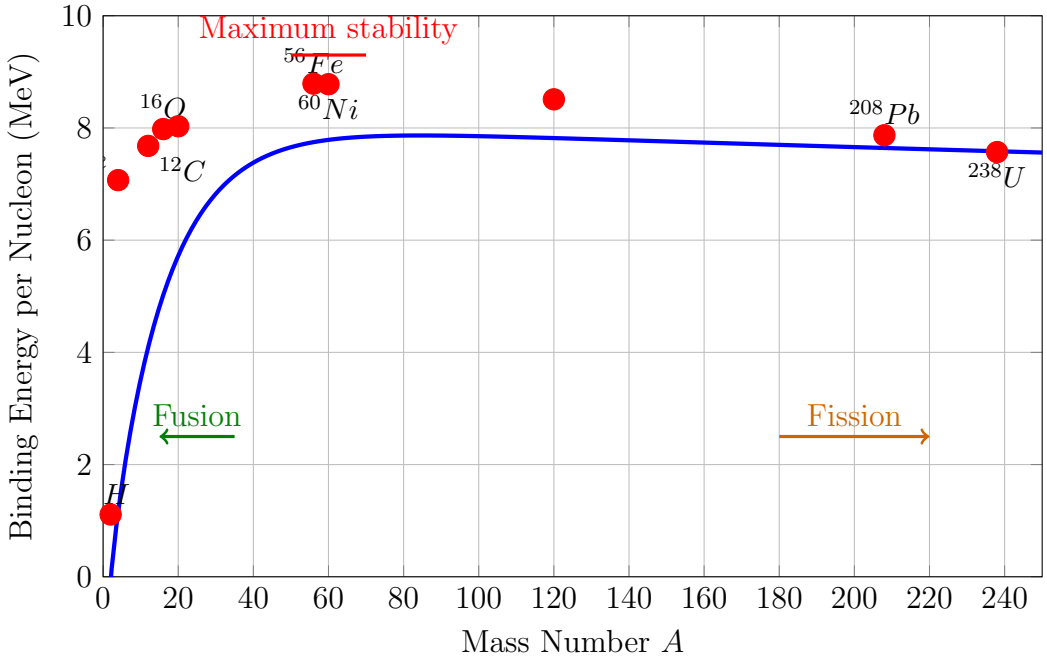


Figure 2.4: Binding energy per nucleon as a function of mass number

2.4.2 Key Features of the Binding Energy Curve

1. **Very Light Nuclei** ($A < 20$):
   - Binding energy per nucleon increases rapidly with A
   - ^2H (deuteron) has very low $BE/A \approx 1.1$ MeV
   - ^4He shows anomalously high binding ($BE/A \approx 7.1$ MeV)
   - Other light nuclei with A multiple of 4 also show peaks (^{12}C, ^{16}O)
2. **Maximum Binding Region** ($A \approx 50 - 70$):
   - Maximum occurs around ^{56}Fe and ^{62}Ni with $BE/A \approx 8.8$ MeV
   - These are the most stable nuclei
   - Iron group elements are the end products of stellar nucleosynthesis
3. **Heavy Nuclei** ($A > 70$):
   - Binding energy per nucleon slowly decreases
   - $BE/A \approx 7.6$ MeV for uranium
   - Decrease due to increasing Coulomb repulsion

2.4.3 Energy Release in Nuclear Reactions

The shape of the binding energy curve explains the energy release in fusion and fission:

Nuclear Fusion

Light nuclei can combine (fuse) to form heavier nuclei with higher BE/A, releasing energy:

$$\text{Example:} \quad {}^{2}H + {}^{3}H \rightarrow {}^{4}He + n + 17.6 \text{ MeV} \tag{2.23}$$

Energy released per nucleon involved:

$$Q/A = \frac{17.6 \text{ MeV}}{5} = 3.52 \text{ MeV/nucleon}$$

Nuclear Fission

Heavy nuclei can split into lighter fragments with higher BE/A, releasing energy:

$$\text{Example:} \quad {}^{235}U + n \rightarrow {}^{141}Ba + {}^{92}Kr + 3n + 200 \text{ MeV} \tag{2.24}$$

Energy released per nucleon:

$$Q/A = \frac{200 \text{ MeV}}{236} \approx 0.85 \text{ MeV/nucleon}$$

Note: Fusion vs. Fission

Although fission releases more total energy per reaction, fusion releases more energy *per unit mass*:

- Fusion: $\sim 3 - 7$ MeV per nucleon
- Fission: $\sim 0.8 - 1$ MeV per nucleon

This is why hydrogen bombs are more powerful than fission bombs of the same mass.

2.5 Packing Fraction

The packing fraction is an older concept that was historically used to characterize nuclear masses before binding energies became the standard.

Definition 2.5.1. The **packing fraction** (f) is defined as:

$$f = \frac{M - A}{A} \tag{2.25}$$

where M is the atomic mass in atomic mass units and A is the mass number.

Alternatively, it can be expressed as:

$$f = \frac{M - A}{A} \times 10^4 \quad \text{(in units of } 10^{-4}\text{)} \tag{2.26}$$

2.5.1 Physical Interpretation

The packing fraction measures how tightly nucleons are "packed" in the nucleus:

- If nucleons had exactly 1 u mass and no binding energy: $f = 0$
- Negative f indicates mass defect (binding)
- More negative f means more tightly bound nucleus

2.5.2 Relation to Binding Energy

The packing fraction is related to the binding energy per nucleon:

$$\frac{BE}{A} = -f \times 931.494 \text{ MeV} \tag{2.27}$$

(approximately, ignoring the neutron-proton mass difference)

Example 2.5.1. Calculating Packing Fraction

Calculate the packing fraction for ^{56}Fe with atomic mass 55.934937 u.

Solution:

$$f = \frac{M - A}{A} = \frac{55.934937 - 56}{56}$$
$$= \frac{-0.065063}{56} = -0.001162$$

In conventional units:

$$f = -0.001162 \times 10^4 = -11.62 \text{ (in units of } 10^{-4})$$

The negative packing fraction indicates significant binding energy.

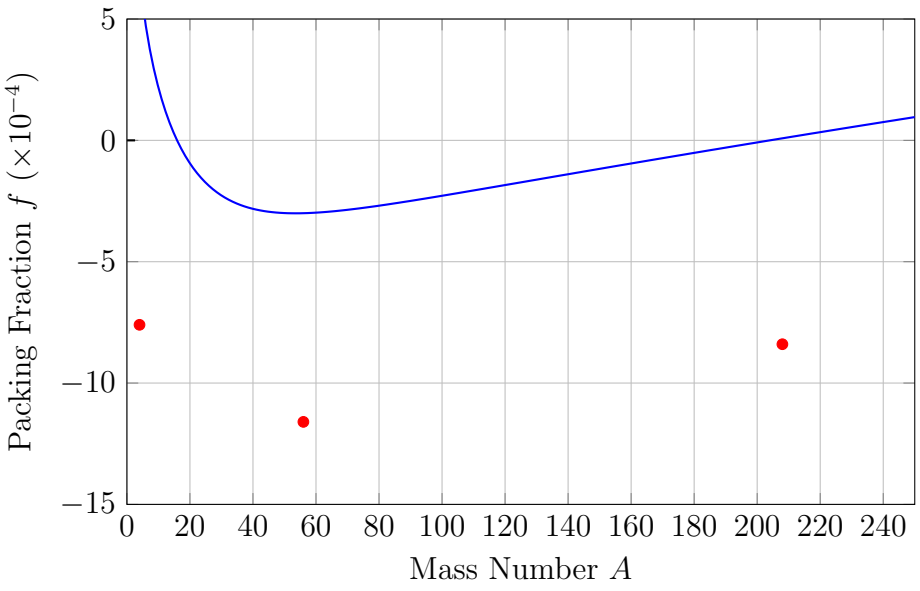


Figure 2.5: Packing fraction as a function of mass number

2.6 Semi-Empirical Mass Formula (SEMF)

2.6.1 Introduction

The **Semi-Empirical Mass Formula** (SEMF), also known as the Bethe-Weizsäcker formula, provides a way to estimate nuclear masses and binding energies based on a liquid drop model of the nucleus.

The formula is "semi-empirical" because it combines theoretical understanding of nuclear forces with empirically determined constants.

2.6.2 The Liquid Drop Model

The SEMF is based on the analogy between a nucleus and a liquid drop:

- Nucleons are like molecules in a liquid
- Short-range nuclear force is like intermolecular forces
- Constant nuclear density is like incompressible liquid
- Surface tension effects at nuclear boundary

2.6.3 The Binding Energy Formula

The SEMF expresses the binding energy as a sum of five terms:

$$\boxed{BE = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta(A,Z)} \tag{2.28}$$

where the constants are approximately:

$$a_V \approx 15.75 \text{ MeV} \quad \text{(volume term)} \tag{2.29}$$
$$a_S \approx 17.8 \text{ MeV} \quad \text{(surface term)} \tag{2.30}$$
$$a_C \approx 0.711 \text{ MeV} \quad \text{(Coulomb term)} \tag{2.31}$$
$$a_A \approx 23.7 \text{ MeV} \quad \text{(asymmetry term)} \tag{2.32}$$

The pairing term $\delta(A,Z)$ is:

$$\delta(A,Z) = \begin{cases} +a_P A^{-1/2} & \text{even-even nuclei} \\ 0 & \text{odd-A nuclei} \\ -a_P A^{-1/2} & \text{odd-odd nuclei} \end{cases} \tag{2.33}$$

where $a_P \approx 11.18$ MeV.

2.6.4 Physical Interpretation of Each Term

1. Volume Term: $a_V A$

Volume Term

Physical basis: Each nucleon interacts with its neighbors via the strong force.
Contribution: Proportional to the number of nucleons.
Sign: Positive (increases binding energy)
This term assumes that each nucleon contributes equally to the binding energy, giving $BE \propto A$. This would be exact if all nucleons were in the interior of the nucleus.

2. Surface Term: $-a_S A^{2/3}$

Surface Term

Physical basis: Nucleons on the surface have fewer neighbors and are less tightly bound.
Contribution: Proportional to surface area $\propto R^2 \propto A^{2/3}$.
Sign: Negative (decreases binding energy)
The surface area is $4\pi R^2 = 4\pi R_0^2 A^{2/3}$, so the correction to binding energy is proportional to $A^{2/3}$.

3. Coulomb Term: $-a_C \frac{Z^2}{A^{1/3}}$

Coulomb Term

Physical basis: Electrostatic repulsion between protons reduces binding.
Contribution: The Coulomb energy of a uniformly charged sphere is:

$$E_C = \frac{3}{5}\frac{k(Ze)^2}{R} = \frac{3}{5}\frac{k(Ze)^2}{R_0 A^{1/3}} \tag{2.34}$$

Sign: Negative (decreases binding energy)
This term explains why heavy nuclei are less stable—Coulomb repulsion becomes increasingly important.

4. Asymmetry Term: $-a_A \frac{(A-2Z)^2}{A}$

Asymmetry Term

Physical basis: Nuclei are most stable when $N \approx Z$ (for light nuclei).
Since $N = A - Z$, the neutron excess is:

$$N - Z = A - 2Z \tag{2.35}$$

Contribution: Deviation from $N = Z$ decreases binding energy.
Sign: Negative (decreases binding energy)
This term arises from the Pauli exclusion principle and the filling of energy levels.

5. Pairing Term: $\delta(A, Z)$

Pairing Term

Physical basis: Nucleons pair up with antiparallel spins.
Observations:

- Even-even nuclei (even Z, even N) are more stable
- Odd-odd nuclei (odd Z, odd N) are less stable
- Odd-A nuclei are intermediate

The term is proportional to $A^{-1/2}$ because the pairing effect is more pronounced in small nuclei.

2.6.5 Alternative Form: Mass Formula

The SEMF can also be expressed in terms of atomic mass:

$$M(A, Z) = Zm_H + Nm_n - \frac{BE(A, Z)}{c^2} \tag{2.36}$$

where $BE(A, Z)$ is given by the semi-empirical formula above.

Example 2.6.1. Calculating Binding Energy Using SEMF

Calculate the binding energy of ^{56}Fe using the semi-empirical mass formula and compare with the experimental value.

Given: $A = 56$, $Z = 26$, $N = 30$
Solution:
Calculate each term:
1. Volume term:

$$a_V A = 15.75 \times 56 = 882.0 \text{ MeV}$$

2. Surface term:

$$a_S A^{2/3} = 17.8 \times 56^{2/3} = 17.8 \times 14.64 = 260.6 \text{ MeV}$$

3. Coulomb term:

$$a_C \frac{Z^2}{A^{1/3}} = 0.711 \times \frac{26^2}{56^{1/3}} = 0.711 \times \frac{676}{3.826} = 125.7 \text{ MeV}$$

4. Asymmetry term:

$$a_A \frac{(A-2Z)^2}{A} = 23.7 \times \frac{(56-52)^2}{56} = 23.7 \times \frac{16}{56} = 6.8 \text{ MeV}$$

5. Pairing term (even-even nucleus):

$$\delta = +a_P A^{-1/2} = 11.18 \times 56^{-1/2} = 11.18 \times 0.134 = 1.5 \text{ MeV}$$

Total binding energy:

$$\begin{aligned} BE &= 882.0 - 260.6 - 125.7 - 6.8 + 1.5 \\ &= 490.4 \text{ MeV} \end{aligned}$$

Experimental value: $BE = 492.3$ MeV
Error: $\frac{|492.3-490.4|}{492.3} \times 100\% = 0.4\%$
The SEMF gives excellent agreement with experiment!

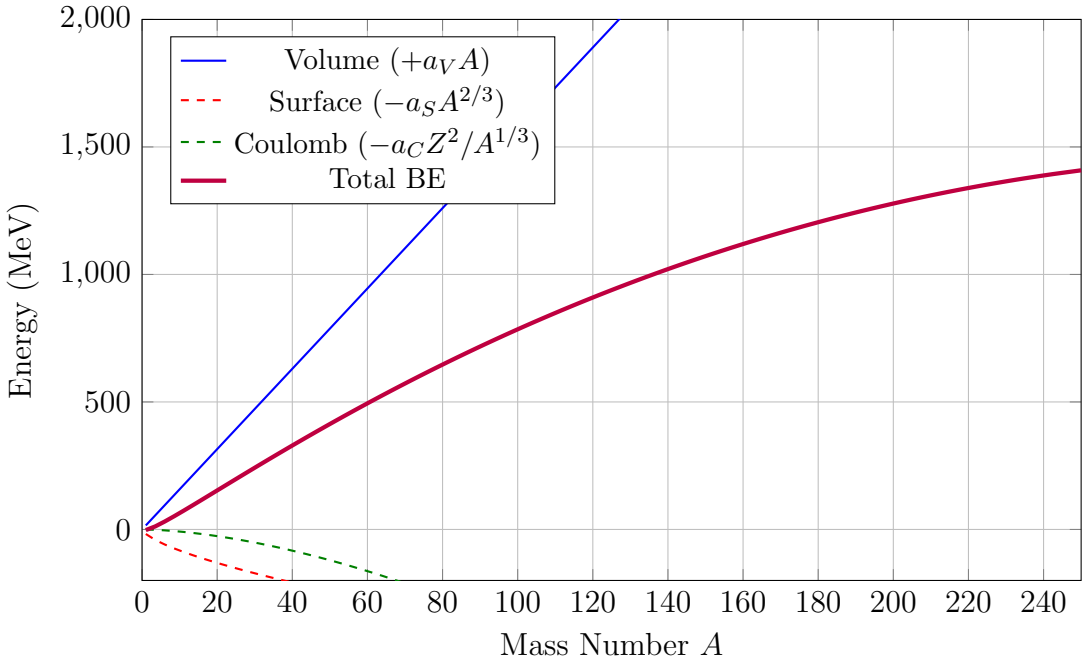


Figure 2.6: Contributions of different terms in the SEMF to total binding energy

2.6.6 Applications of SEMF

The semi-empirical mass formula has several important applications:

1. **Predicting unknown masses**: Estimate masses of unstable or undiscovered nuclei
2. **Q-values of reactions**: Calculate energy release in nuclear reactions
3. **Stability predictions**: Determine most stable isobar for given A
4. **Beta decay systematics**: Predict decay modes and energies
5. **Fission barrier heights**: Estimate activation energy for fission

2.7 Nuclear Stability

2.7.1 The Line of Stability

Not all combinations of protons and neutrons form stable nuclei. Stable nuclei lie along a curve in the (N, Z) plane called the **line of stability** or **valley of stability**.

Stability Trends

1. **Light Nuclei ($A < 40$)**: Stable nuclei have $N \approx Z$

    - Example: ^{12}C has $Z = 6$, $N = 6$
    - Example: ^{16}O has $Z = 8$, $N = 8$

2. **Heavy Nuclei ($A > 40$)**: Stable nuclei have $N > Z$

    - The ratio N/Z increases with A
    - Example: ^{208}Pb has $Z = 82$, $N = 126$, $N/Z = 1.54$

3. **Very Heavy Nuclei ($Z > 83$)**: No stable nuclei

    - All nuclei with $Z > 83$ are radioactive
    - Coulomb repulsion overcomes nuclear binding

2.7.2 Why Does N/Z Increase for Heavy Nuclei?

For light nuclei, having equal numbers of protons and neutrons ($N = Z$) minimizes the asymmetry term in the SEMF. However, for heavy nuclei:

- Coulomb repulsion between protons increases as Z^2
- Additional neutrons provide more nuclear binding without adding Coulomb repulsion
- The optimal ratio shifts to $N > Z$ to balance these competing effects

2.7.3 Determining Stability Using SEMF

For a given mass number A, we can find the most stable value of Z by minimizing the mass (or maximizing the binding energy).

Taking the derivative of the binding energy with respect to Z and setting it to zero:

$$\frac{\partial BE}{\partial Z} = 0 \tag{2.37}$$

This gives:

$$Z_{\text{stable}} = \frac{A}{2 + a_C A^{2/3}/(2a_A)} \tag{2.38}$$

For light nuclei (A small): $Z \approx A/2$, giving $N \approx Z$

For heavy nuclei (A large): $Z < A/2$, giving $N > Z$

2.7.4 Magic Numbers

Certain numbers of protons or neutrons result in particularly stable configurations, called **magic numbers**:

$$\text{Magic numbers: } 2, 8, 20, 28, 50, 82, 126 \tag{2.39}$$

Magic Numbers

Nuclei with magic numbers of protons or neutrons are:

- More tightly bound than neighbors
- More abundant in nature
- Have more stable isotopes
- Have lower neutron capture cross sections

Doubly magic nuclei (both Z and N magic) are especially stable:

- ^4He ($Z = 2$, $N = 2$)
- ^{16}O ($Z = 8$, $N = 8$)
- ^{40}Ca ($Z = 20$, $N = 20$)
- ^{48}Ca ($Z = 20$, $N = 28$)
- ^{208}Pb ($Z = 82$, $N = 126$)

Magic numbers arise from the *nuclear shell model*, analogous to electron shells in atoms. Large energy gaps occur at these nucleon numbers, similar to the noble gas configurations.

2.7.5 Beta Stability and Decay Modes

Nuclei away from the stability line undergo radioactive decay to reach stability:

1. **Beta-minus (β^-) decay**: For neutron-rich nuclei ($N > N_{\text{stable}}$)

$$n \rightarrow p + e^- + \bar{\nu}_e \tag{2.40}$$

   Effect: Z increases by 1, N decreases by 1, A unchanged

2. **Beta-plus (β^+) decay or Electron Capture**: For proton-rich nuclei ($Z > Z_{\text{stable}}$)

$$p \rightarrow n + e^+ + \nu_e \tag{2.41}$$

   Effect: Z decreases by 1, N increases by 1, A unchanged

3. **Alpha decay**: For very heavy nuclei ($Z > 83$)

$$^A_Z X \rightarrow ^{A-4}_{Z-2} Y + ^4_2 He \tag{2.42}$$

   Effect: Z decreases by 2, N decreases by 2, A decreases by 4

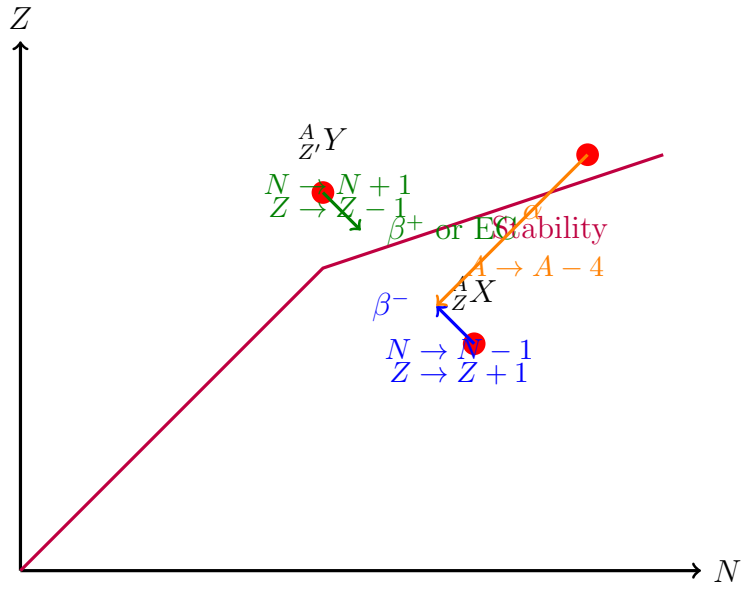


Figure 2.8: Decay modes relative to the line of stability

2.8 Summary

In this chapter, we have developed a comprehensive understanding of basic nuclear properties:

1. **Nuclear Composition**: Nuclei consist of Z protons and N neutrons, with mass number $A = Z + N$. Isotopes, isotones, and isobars are different classifications of nuclei.

2. **Nuclear Size and Density**: Nuclear radius follows $R = R_0 A^{1/3}$ with $R_0 \approx 1.2$ fm. Nuclear density is approximately constant at $\rho \approx 2.3 \times 10^{17}$ kg/m^3.

3. **Mass Defect and Binding Energy**: The mass of a nucleus is less than the sum of its constituents. The mass defect ΔM corresponds to the binding energy via $BE = \Delta M c^2$.

4. **Binding Energy Curve**: The binding energy per nucleon peaks around ${}^{56}Fe$ at about 8.8 MeV/nucleon. This explains energy release in fusion (light nuclei) and fission (heavy nuclei).

5. **Semi-Empirical Mass Formula**: The SEMF provides a quantitative model for nuclear masses based on volume, surface, Coulomb, asymmetry, and pairing terms.

6. **Nuclear Stability**: Stable nuclei lie along a line in the (N, Z) plane. Light nuclei have $N \approx Z$, while heavy nuclei have $N > Z$. Magic numbers indicate particularly stable configurations.

These fundamental properties provide the foundation for understanding nuclear reactions, radioactive decay, and nuclear energy applications, which will be explored in subsequent chapters.

2.9 Problems

1. Calculate the nuclear radius and density for:
   (a) ^{27}Al
   (b) ^{120}Sn
   (c) ^{197}Au
2. The atomic mass of ^{16}O is 15.994915 u. Calculate:
   (a) The mass defect
   (b) The total binding energy
   (c) The binding energy per nucleon
3. Using the semi-empirical mass formula, calculate the binding energy of:
   (a) ^{4}He
   (b) ^{12}C
   (c) ^{238}U

   Compare with experimental values.
4. For the isobaric chain with $A = 100$, use the SEMF to determine the most stable value of Z.
5. Calculate the packing fraction for ^{208}Pb (mass = 207.976627 u).
6. Show that the Coulomb energy of a uniformly charged sphere of charge Q and radius R is:
$$E_C = \frac{3}{5}\frac{kQ^2}{R}$$
7. Estimate the energy released when four ^{1}H nuclei fuse to form ^{4}He. Compare this to the energy from burning an equivalent mass of gasoline.
8. A nucleus has a binding energy per nucleon of 7.5 MeV. If it splits into two equal fragments each with binding energy per nucleon of 8.2 MeV, how much energy is released?
9. Using the SEMF, determine which nucleus has greater binding energy: $^{100}_{44}Ru$ or $^{100}_{45}Rh$?
10. The nuclear radius is measured to be 5.0 fm. Estimate the mass number A of this nucleus.
11. Show that for the asymmetry term to be minimized, we need $N = Z$ for light nuclei.
12. Calculate the ratio of Coulomb energy to total binding energy for:
    (a) ^{12}C
    (b) ^{56}Fe
    (c) ^{238}U

    What trend do you observe?

2.10 Further Reading

1. Krane, K.S., *Introductory Nuclear Physics*, John Wiley & Sons (1987) - Chapter 3

2. Lilley, J., *Nuclear Physics: Principles and Applications*, John Wiley & Sons (2001) - Chapters 2-3

3. Wong, S.S.M., *Introductory Nuclear Physics*, Wiley-VCH (1998) - Chapter 2

4. Cottingham, W.N. and Greenwood, D.A., *An Introduction to Nuclear Physics*, Cambridge University Press (2001) - Chapters 3-4

5. Povh, B., et al., *Particles and Nuclei: An Introduction to the Physical Concepts*, Springer (2008) - Chapter 2

6. Basdevant, J.-L., Rich, J., and Spiro, M., *Fundamentals in Nuclear Physics*, Springer (2005) - Chapter 3

Chapter 3

Radioactivity

3.1 Introduction

Radioactivity is one of the most remarkable phenomena in nuclear physics, representing the spontaneous transformation of unstable atomic nuclei into more stable configurations through the emission of particles and energy. This natural process, discovered by Henri Becquerel in 1896, revolutionized our understanding of atomic structure and has led to countless applications in medicine, industry, energy production, and scientific research.

The discovery of radioactivity began when Becquerel observed that uranium salts could expose photographic plates even when wrapped in black paper, indicating the emission of penetrating radiation. Marie and Pierre Curie extended this work by discovering new radioactive elements—polonium and radium—and introducing the term "radioactivity." Ernest Rutherford and Frederick Soddy later proposed that radioactivity involves the transformation of one element into another, a revolutionary concept at the time.

Radioactive decay is a quantum mechanical process that occurs because many nuclear configurations are unstable. These unstable nuclei, called **radionuclides**, spontaneously undergo transformations to reach more stable states by emitting particles such as alpha particles, beta particles, or gamma rays. The process is governed by fundamental laws that allow us to predict decay rates, calculate half-lives, and understand nuclear stability.

In this chapter, we will explore:

- The three primary types of radioactive decay: alpha (α), beta (β), and gamma (γ)
- The mathematical laws governing radioactive decay
- Quantitative measures: decay constant, half-life, mean life, and activity
- Successive decay chains and equilibrium conditions
- Artificial radioactivity and nuclear transmutation
- Practical applications of radioisotopes in various fields

Understanding radioactivity is essential not only for nuclear physics but also for applications ranging from cancer treatment to carbon dating, from nuclear power generation to smoke detectors in our homes.

3.2 Types of Radioactive Decay

Radioactive decay can occur through several mechanisms, each characterized by different emitted particles and energy transformations. The three principal types are alpha decay, beta decay, and gamma decay.

3.2.1 Alpha Decay (α-Decay)

Alpha decay is a process in which an unstable nucleus emits an alpha particle, which consists of two protons and two neutrons (identical to a helium-4 nucleus).

The Alpha Decay Process

In alpha decay, the parent nucleus transforms according to:

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\mathrm{He} + Q_\alpha \tag{3.1} $$

where:

- X is the parent nucleus with mass number A and atomic number Z
- Y is the daughter nucleus
- ${}^{4}_{2}\mathrm{He}$ is the alpha particle (often denoted as α)
- Q_α is the energy released (Q-value)

Key Characteristics of Alpha Decay

1. The parent nucleus loses 4 nucleons (mass number decreases by 4)
2. The atomic number decreases by 2 (element changes)
3. Alpha particles are helium-4 nuclei: ${}^{4}_{2}\mathrm{He}^{2+}$
4. Typical energies: 4-9 MeV
5. Short range in matter (few cm in air, < 0.1 mm in tissue)
6. Common in heavy nuclei ($A > 200$, $Z > 82$)

Example 3.2.1. Alpha Decay of Uranium-238

The most abundant uranium isotope undergoes alpha decay:

$$ {}^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He} + 4.27\ \mathrm{MeV} $$

Uranium-238 (92 protons) decays to thorium-234 (90 protons), releasing an alpha particle with kinetic energy of approximately 4.2 MeV.

Energy Considerations in Alpha Decay

The Q-value (energy released) in alpha decay is calculated from the mass difference:

$$Q_\alpha = [M(X) - M(Y) - M(\alpha)]c^2 \tag{3.2}$$

For alpha decay to be energetically possible (spontaneous), we require $Q_\alpha > 0$.

The released energy is shared between the alpha particle and the recoiling daughter nucleus according to conservation of momentum and energy:

$$K_\alpha = \frac{A-4}{A} Q_\alpha \quad \text{and} \quad K_Y = \frac{4}{A} Q_\alpha \tag{3.3}$$

where K_α and K_Y are the kinetic energies of the alpha particle and daughter nucleus, respectively.

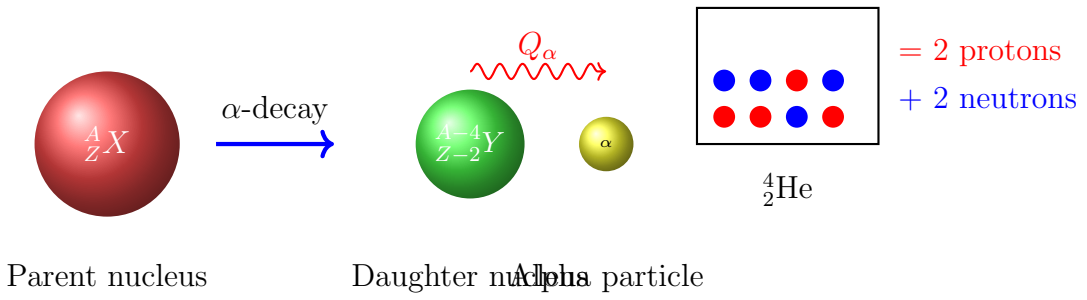


Figure 3.1: Schematic representation of alpha decay

Quantum Tunneling and the Geiger-Nuttall Law

Alpha decay occurs through quantum mechanical tunneling. The alpha particle must penetrate the Coulomb barrier of the nucleus, even though classically it has insufficient energy.

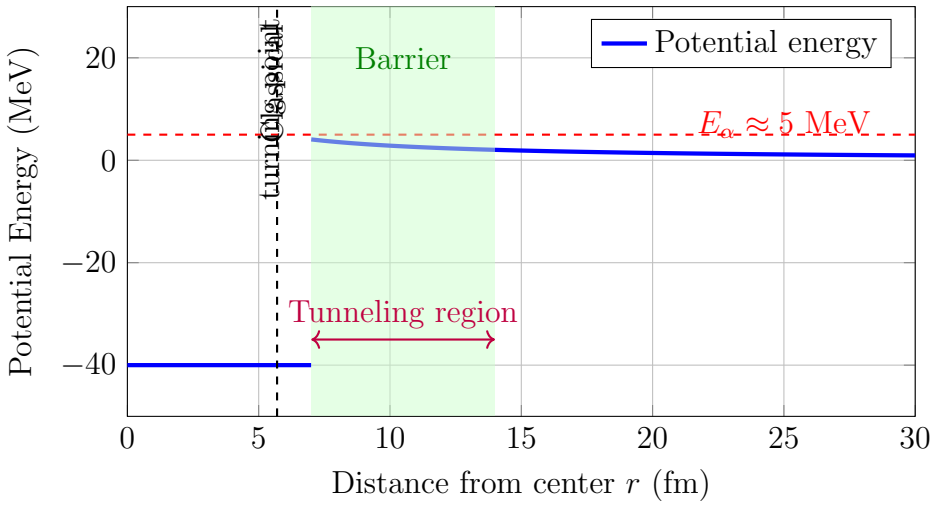


Figure 3.2: Potential energy barrier for alpha decay showing quantum tunneling

The probability of tunneling depends exponentially on the barrier width and height, leading to the **Geiger-Nuttall law**:

$$\log_{10} \lambda = A + B \log_{10} E_\alpha \tag{3.4}$$

where λ is the decay constant, E_α is the alpha particle energy, and A and B are constants for a given decay series.

This relationship explains why small changes in alpha particle energy lead to dramatic changes in half-life.

3.2.2 Beta Decay (β-Decay)

Beta decay involves the transformation of a neutron into a proton (or vice versa) within the nucleus, accompanied by the emission of electrons or positrons and neutrinos.

Beta-Minus Decay (β^-)

In beta-minus decay, a neutron converts to a proton:

$$n \to p + e^- + \bar{\nu}_e \tag{3.5}$$

The nuclear reaction is:

$${}^{A}_{Z}X \to {}^{A}_{Z+1}Y + e^- + \bar{\nu}_e + Q_{\beta^-} \tag{3.6}$$

Beta-Minus Decay Characteristics

- Mass number A remains unchanged
- Atomic number Z increases by 1
- An electron (e^- or β^-) is emitted
- An electron antineutrino ($\bar{\nu}_e$) is emitted
- Occurs in neutron-rich nuclei
- Continuous energy spectrum of electrons

Example 3.2.2. Beta-Minus Decay of Carbon-14

$${}^{14}_{6}\mathrm{C} \to {}^{14}_{7}\mathrm{N} + e^- + \bar{\nu}_e + 0.156 \text{ MeV}$$

Carbon-14 decays to nitrogen-14. This process is the basis for radiocarbon dating.

Beta-Plus Decay (β^+) and Electron Capture

In beta-plus decay, a proton converts to a neutron:

$$p \to n + e^+ + \nu_e \tag{3.7}$$

The nuclear reaction is:

$${}^{A}_{Z}X \to {}^{A}_{Z-1}Y + e^+ + \nu_e + Q_{\beta^+} \tag{3.8}$$

Alternatively, **electron capture** (EC) can occur:

$$ {}^{A}_{Z}X + e^{-} \rightarrow {}^{A}_{Z-1}Y + \nu_e + Q_{EC} \tag{3.9} $$

Beta-Plus Decay and Electron Capture

- Mass number A remains unchanged
- Atomic number Z decreases by 1
- β^+: positron and neutrino emitted
- EC: inner shell electron captured, neutrino and X-rays emitted
- Occurs in proton-rich nuclei

Example 3.2.3. Beta-Plus Decay of Fluorine-18

$$ {}^{18}_{9}\mathrm{F} \rightarrow {}^{18}_{8}\mathrm{O} + e^{+} + \nu_e + 0.634 \text{ MeV} $$

Fluorine-18 is widely used in PET (Positron Emission Tomography) medical imaging.

Energy Distribution in Beta Decay

Unlike alpha particles, which have discrete energies, beta particles show a continuous energy spectrum. This is because the decay energy is shared among three particles (electron/positron, neutrino, and recoiling nucleus).

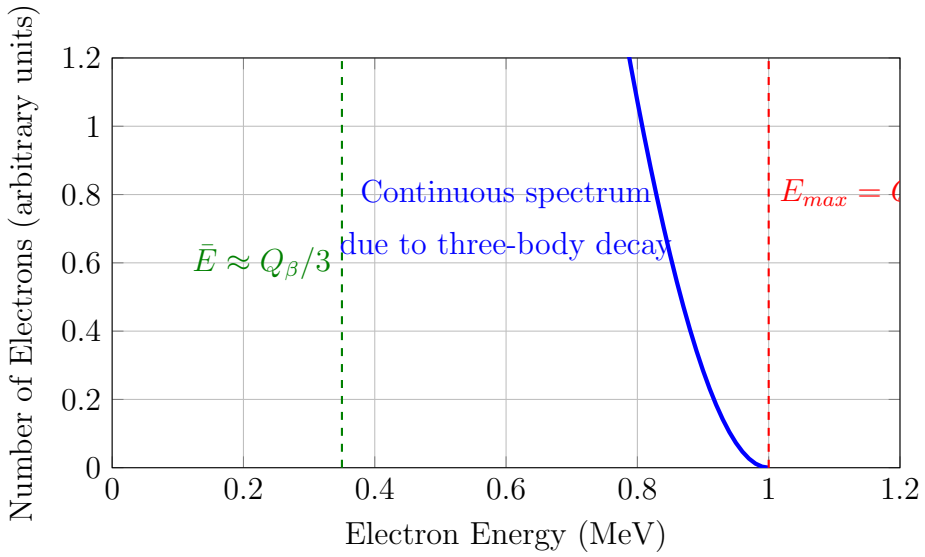


Figure 3.3: Continuous energy spectrum of beta particles

The continuous spectrum initially puzzled physicists and led Wolfgang Pauli to propose the existence of the neutrino in 1930 to conserve energy and momentum.

Fermi Theory of Beta Decay

Enrico Fermi developed the first successful theory of beta decay in 1934, treating it as a weak interaction process. The decay rate is given by:

$$\lambda = \frac{G_F^2}{2\pi^3\hbar^7}|M_{fi}|^2 f(Z, Q_\beta) \tag{3.10}$$

where:

- G_F is the Fermi coupling constant
- $|M_{fi}|^2$ is the nuclear matrix element
- $f(Z, Q_\beta)$ is the Fermi integral (phase space factor)

3.2.3 Gamma Decay (γ-Decay)

Gamma decay involves the emission of high-energy photons (gamma rays) from excited nuclear states.

The Gamma Decay Process

Unlike alpha and beta decay, gamma decay doesn't change the number of protons or neutrons:

$${}^A_Z X^* \rightarrow {}^A_Z X + \gamma \tag{3.11}$$

where X^* denotes an excited nuclear state.

Gamma Decay Characteristics

- No change in A or Z (same nucleus, different energy state)
- Photon emission with discrete energies
- Typical energies: 0.1-10 MeV
- Highly penetrating radiation
- Often follows alpha or beta decay
- Competes with internal conversion

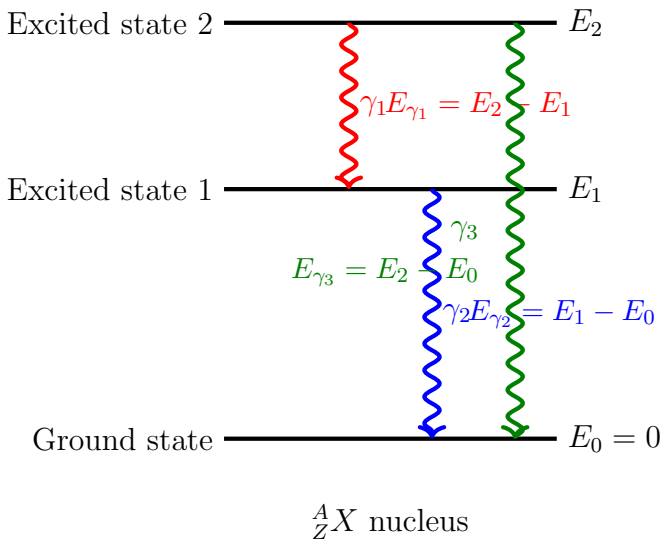


Figure 3.4: Energy level diagram showing gamma decay transitions

Selection Rules and Multipole Radiation

Gamma emission obeys selection rules based on conservation of angular momentum and parity. The emitted radiation is classified by its multipole character:

- **Electric dipole (E1)**: $\Delta L = 1$, parity change
- **Magnetic dipole (M1)**: $\Delta L = 1$, no parity change
- **Electric quadrupole (E2)**: $\Delta L = 2$, no parity change

Higher multipoles are increasingly suppressed, making E1 transitions the fastest.

Internal Conversion

An alternative to gamma emission is **internal conversion**, where the nuclear excitation energy is transferred directly to an inner shell electron, ejecting it from the atom.

$$ {}^A_Z X^* \rightarrow {}^A_Z X + e^-_{IC} \tag{3.12} $$

The internal conversion coefficient is:

$$ \alpha_{IC} = \frac{\text{rate of electron emission}}{\text{rate of gamma emission}} \tag{3.13} $$

Example 3.2.4. Gamma Decay of Cobalt-60

Cobalt-60 undergoes beta decay to excited states of Nickel-60, which then emit gamma rays:

$$ {}^{60}_{27}\text{Co} \rightarrow {}^{60}_{28}\text{Ni}^* + e^- + \bar{\nu}_e $$
$$ {}^{60}_{28}\text{Ni}^* \rightarrow {}^{60}_{28}\text{Ni} + \gamma_1\ (1.17\ \text{MeV}) + \gamma_2\ (1.33\ \text{MeV}) $$

These high-energy gamma rays make Co-60 useful for cancer radiotherapy.

3.2.4 Comparison of Decay Types

Property	**Alpha (α)**	**Beta (β)**	**Gamma (γ)**
Nature	$^4_2$He nucleus	Electron/positron	Photon
Charge	+2e	$\pm$e	0
Mass	4 u	$\sim$0.0005 u	0
Change in A	-4	0	0
Change in Z	-2	±1	0
Typical energy	4-9 MeV	0-3 MeV	0.1-10 MeV
Energy spectrum	Discrete	Continuous	Discrete
Penetration (air)	$\sim$5 cm	$\sim$1 m	$>$100 m
Stopped by	Paper	Aluminum (mm)	Lead (cm)
Ionization	High	Medium	Low

Table 3.1: Comparison of the three main types of radioactive decay

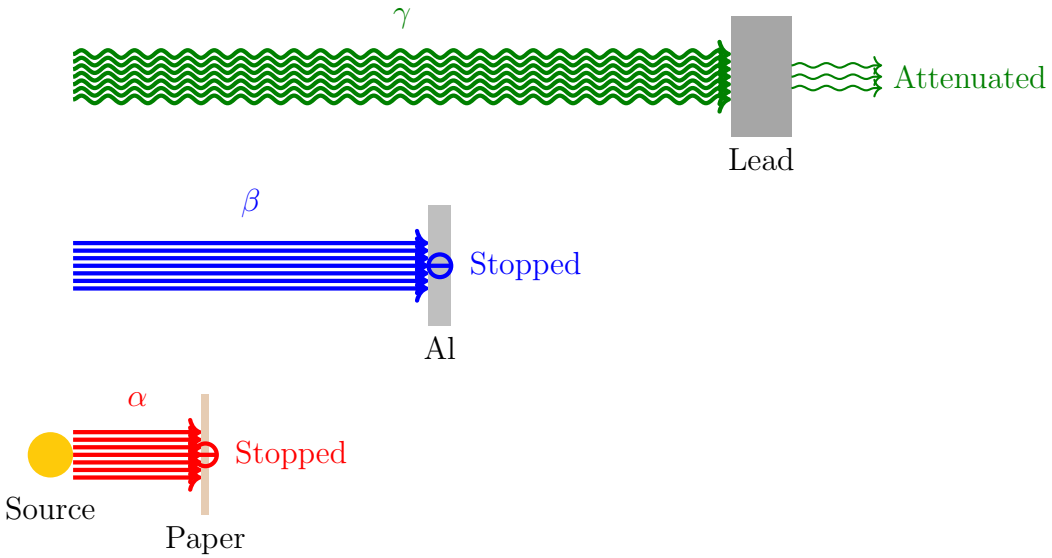


Figure 3.5: Penetrating power of different types of radiation

3.3 Laws of Radioactive Decay

3.3.1 The Radioactive Decay Law

Radioactive decay is a statistical process. We cannot predict when a particular nucleus will decay, but we can predict the average behavior of a large number of nuclei.

Fundamental Assumptions

The mathematical description of radioactive decay is based on two key assumptions:

1. The probability that a nucleus will decay in a given time interval is constant (independent of the nucleus's history or age)

2. The decay of one nucleus is independent of the decay of other nuclei

Derivation of the Decay Law

Derivation of Exponential Decay Law

Let $N(t)$ be the number of radioactive nuclei at time t.
The probability that a nucleus decays in a small time interval dt is proportional to dt:

$$\text{Decay probability} = \lambda\, dt \tag{3.14}$$

where λ is the **decay constant** (probability of decay per unit time).
The number of nuclei that decay in time dt is:

$$dN = -\lambda N\, dt \tag{3.15}$$

The negative sign indicates that N decreases with time.
Rearranging:

$$\frac{dN}{N} = -\lambda\, dt \tag{3.16}$$

Integrating both sides:

$$\int_{N_0}^{N} \frac{dN'}{N'} = -\lambda \int_0^t dt' \tag{3.17}$$

$$\ln N - \ln N_0 = -\lambda t \tag{3.18}$$

$$\ln \frac{N}{N_0} = -\lambda t \tag{3.19}$$

Taking the exponential of both sides:

$$\boxed{N(t) = N_0 e^{-\lambda t}} \tag{3.20}$$

This is the **radioactive decay law**.

Law 3.3.1 (Radioactive Decay Law)**.** *The number of radioactive nuclei decreases exponentially with time according to:*

$$N(t) = N_0 e^{-\lambda t} \tag{3.21}$$

where:

- $N(t)$ *is the number of nuclei at time* t
- N_0 *is the initial number of nuclei at* $t = 0$
- λ *is the decay constant* (s^{-1})
- e *is Euler's number* (≈ 2.718)

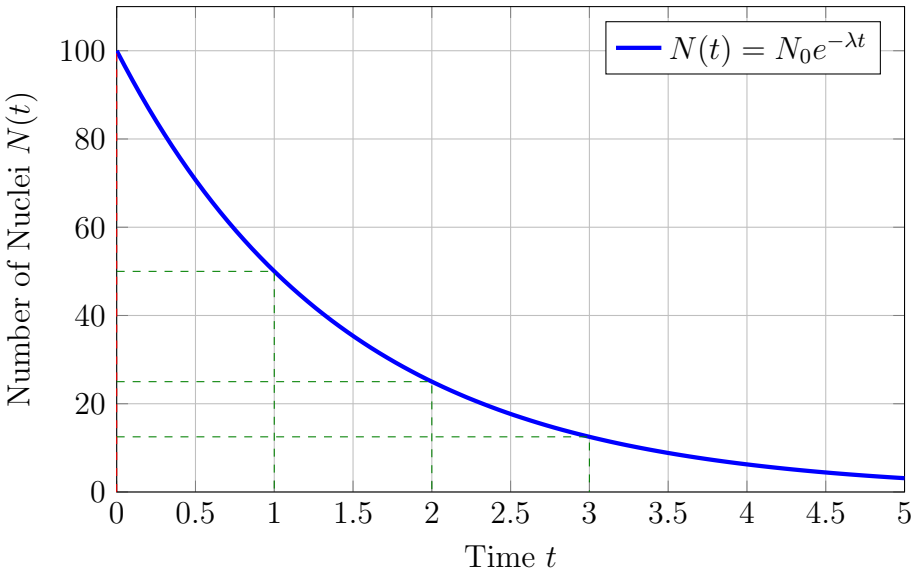


Figure 3.6: Exponential decay showing successive half-lives

3.3.2 Half-Life

Definition 3.3.1. The **half-life** ($t_{1/2}$ or $T_{1/2}$) is the time required for half of the radioactive nuclei to decay.

At $t = t_{1/2}$, we have $N(t_{1/2}) = N_0/2$. Substituting into the decay law:

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}} \tag{3.22}$$

$$\frac{1}{2} = e^{-\lambda t_{1/2}} \tag{3.23}$$

Taking the natural logarithm:

$$-\ln 2 = -\lambda t_{1/2} \tag{3.24}$$

$$\boxed{t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}} \tag{3.25}$$

Properties of Half-Life

1. Half-life is **independent of the number of nuclei** present
2. After n half-lives, the fraction remaining is $(1/2)^n$
3. Half-lives range from microseconds to billions of years
4. Each radioactive isotope has a characteristic half-life

Example 3.3.1. Decay of Iodine-131

Iodine-131, used in medical treatments, has a half-life of 8.0 days.

(a) What is its decay constant?

Solution:

$$\begin{aligned}\lambda &= \frac{0.693}{t_{1/2}} = \frac{0.693}{8.0 \text{ days}} \\ &= 0.0866 \text{ day}^{-1} \\ &= 1.00 \times 10^{-6} \text{ s}^{-1}\end{aligned}$$

(b) If a hospital has 100 mg of I-131, how much remains after 24 days?
Solution:
24 days = 3 half-lives

$$\begin{aligned}N(t) &= N_0 \left(\frac{1}{2}\right)^3 = 100 \text{ mg} \times \frac{1}{8} \\ &= 12.5 \text{ mg}\end{aligned}$$

Alternatively:

$$\begin{aligned}N(t) &= N_0 e^{-\lambda t} = 100 \times e^{-0.0866 \times 24} \\ &= 100 \times e^{-2.078} = 100 \times 0.125 \\ &= 12.5 \text{ mg}\end{aligned}$$

3.3.3 Mean Life (Average Life)

Definition 3.3.2. The **mean life** or **average life** (τ) is the average time that a nucleus exists before decaying.

To find the mean life, we calculate the weighted average:

$$\tau = \frac{\int_0^\infty t \, dN}{\int_0^\infty dN} \tag{3.26}$$

Since $dN = -\lambda N_0 e^{-\lambda t} dt$:

$$\tau = \frac{\int_0^\infty t \lambda N_0 e^{-\lambda t} dt}{N_0} = \lambda \int_0^\infty t e^{-\lambda t} dt \tag{3.27}$$

Using integration by parts:

$$\tau = \lambda \left[-\frac{t}{\lambda} e^{-\lambda t} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} dt \right] \tag{3.28}$$

$$\tau = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda} \tag{3.29}$$

$$\boxed{\tau = \frac{1}{\lambda} = \frac{t_{1/2}}{\ln 2} = 1.443 \, t_{1/2}} \tag{3.30}$$

The mean life is approximately 44% longer than the half-life.

3.3.4 Activity

Definition 3.3.3. The **activity** (A) of a radioactive sample is the rate at which decays occur.

$$A(t) = -\frac{dN}{dt} = \lambda N(t) \tag{3.31}$$

Since $N(t) = N_0 e^{-\lambda t}$:

$$\boxed{A(t) = A_0 e^{-\lambda t}} \tag{3.32}$$

where $A_0 = \lambda N_0$ is the initial activity.

Units of Activity

1. **Becquerel (Bq)**: SI unit

$$1 \text{ Bq} = 1 \text{ decay per second} \tag{3.33}$$

2. **Curie (Ci)**: Traditional unit

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ Bq} \tag{3.34}$$

Originally defined as the activity of 1 gram of radium-226.

Common subdivisions:

- millicurie: 1 mCi $= 10^{-3}$ Ci $= 3.7 \times 10^7$ Bq
- microcurie: 1 μCi $= 10^{-6}$ Ci $= 3.7 \times 10^4$ Bq

Example 3.3.2. Activity Calculation
A sample contains 10^{15} atoms of ^{60}Co ($t_{1/2} = 5.27$ years).
(a) Calculate the decay constant.
Solution:

$$\begin{aligned}\lambda &= \frac{0.693}{t_{1/2}} = \frac{0.693}{5.27 \times 3.156 \times 10^7 \text{ s}} \\ &= 4.17 \times 10^{-9} \text{ s}^{-1}\end{aligned}$$

(b) Calculate the initial activity in Bq and Ci.
Solution:

$$\begin{aligned}A_0 &= \lambda N_0 = 4.17 \times 10^{-9} \times 10^{15} \\ &= 4.17 \times 10^6 \text{ Bq}\end{aligned}$$

$$\begin{aligned}A_0 &= \frac{4.17 \times 10^6}{3.7 \times 10^{10}} \text{ Ci} \\ &= 1.13 \times 10^{-4} \text{ Ci} = 113 \ \mu\text{Ci}\end{aligned}$$

3.3.5 Specific Activity

Definition 3.3.4. The **specific activity** is the activity per unit mass of a radioactive substance.

For a pure radioactive isotope:

$$\text{Specific Activity} = \frac{A}{m} = \frac{\lambda N_A}{M} \tag{3.35}$$

where:

- $N_A = 6.022 \times 10^{23}$ mol^{-1} (Avogadro's number)
- M is the molar mass in g/mol

Example 3.3.3. Specific Activity of Radium-226
Calculate the specific activity of pure ^{226}Ra ($t_{1/2} = 1600$ years).
Solution:

$$\lambda = \frac{0.693}{1600 \times 3.156 \times 10^7} = 1.37 \times 10^{-11} \text{ s}^{-1}$$

For 1 gram ($M = 226$ g/mol):

$$N = \frac{N_A}{M} = \frac{6.022 \times 10^{23}}{226} = 2.67 \times 10^{21} \text{ atoms}$$

$$\begin{aligned} A = \lambda N &= 1.37 \times 10^{-11} \times 2.67 \times 10^{21} \\ &= 3.66 \times 10^{10} \text{ Bq} \approx 1 \text{ Ci} \end{aligned}$$

This is why 1 Curie was originally defined as the activity of 1 gram of radium!

3.4 Successive Decay and Radioactive Equilibrium

Many radioactive isotopes decay into daughter products that are themselves radioactive, leading to decay chains. Understanding these chains is crucial for nuclear medicine, geology, and environmental science.

3.4.1 Successive Decay

Consider a decay chain where nucleus A decays to B, which then decays to C:

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C \tag{3.36}$$

Rate Equations

The rate of change of each species:

For nucleus A:

$$\frac{dN_A}{dt} = -\lambda_A N_A \tag{3.37}$$

Solution:

$$N_A(t) = N_{A0} e^{-\lambda_A t} \tag{3.38}$$

For nucleus B (produced from A, decays to C):

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B \tag{3.39}$$

Substituting $N_A(t)$:

$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A0} e^{-\lambda_A t} \tag{3.40}$$

This is a first-order linear differential equation. The solution (assuming $N_{B0} = 0$) is:

$$\boxed{N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_A} \left(e^{-\lambda_A t} - e^{-\lambda_B t} \right)} \tag{3.41}$$

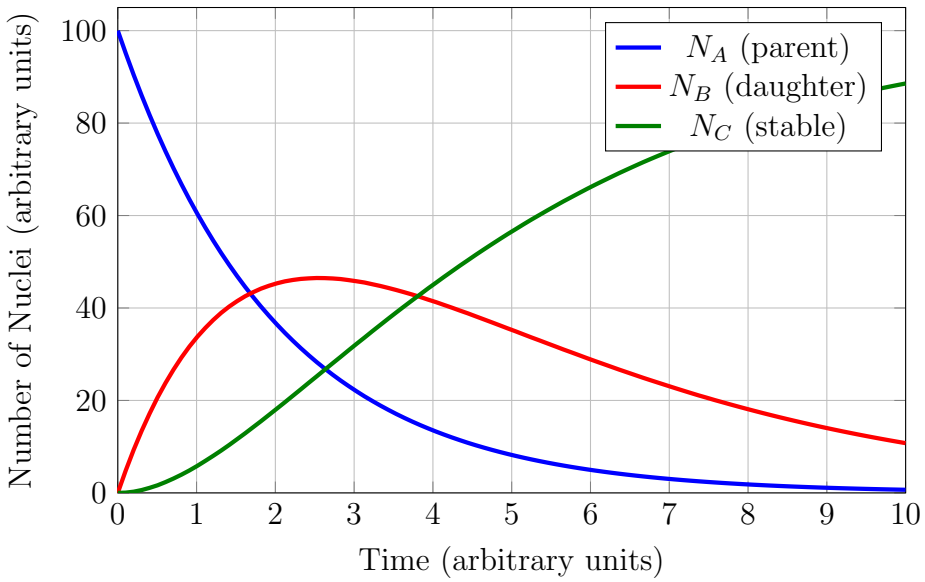


Figure 3.7: Successive decay: parent A, daughter B, and stable product C

3.4.2 Secular Equilibrium

Definition 3.4.1. Secular equilibrium occurs when the parent half-life is much longer than the daughter half-life ($t^A_{1/2} \gg t^B_{1/2}$, or $\lambda_A \ll \lambda_B$).

Under these conditions, after sufficient time:

$$\lambda_A N_A \approx \lambda_B N_B \tag{3.42}$$

This means the activity of the daughter equals the activity of the parent:

$$A_B = A_A \tag{3.43}$$

Characteristics of Secular Equilibrium

1. Parent half-life $\gg$ daughter half-life
2. After equilibrium is reached, activities are equal
3. Daughter decays as fast as it is produced
4. Both decay with the parent's half-life
5. Total activity is twice the parent's activity

Example 3.4.1. Radium-Radon Equilibrium

^{226}Ra ($t_{1/2} = 1600$ years) decays to ^{222}Rn ($t_{1/2} = 3.82$ days).
Since $t_{1/2}^{Ra} \gg t_{1/2}^{Rn}$, secular equilibrium is established.
After about 1 month (several Rn half-lives):

$$A_{Rn} = A_{Ra}$$

A sealed sample of 1 g of Ra will always contain the equilibrium amount of Rn.

3.4.3 Transient Equilibrium

Definition 3.4.2. Transient equilibrium occurs when the parent half-life is moderately longer than the daughter half-life ($t_{1/2}^A > t_{1/2}^B$, but not $\gg$).

At equilibrium:

$$\frac{N_B}{N_A} = \frac{\lambda_A}{\lambda_B - \lambda_A} \tag{3.44}$$

The activity ratio is:

$$\frac{A_B}{A_A} = \frac{\lambda_B}{\lambda_B - \lambda_A} \tag{3.45}$$

Characteristics of Transient Equilibrium

1. Parent half-life $>$ daughter half-life (but not $\gg$)
2. Daughter activity exceeds parent activity
3. Both decay with an apparent half-life close to parent's
4. $A_B/A_A = \lambda_B/(\lambda_B - \lambda_A) > 1$

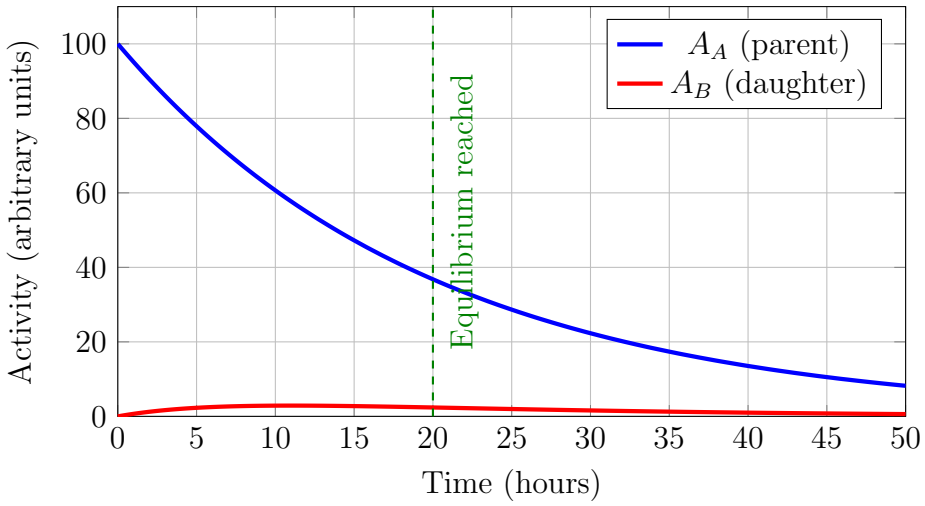


Figure 3.8: Transient equilibrium showing daughter activity exceeding parent

Example 3.4.2. Molybdenum-Technetium Generator

^{99}Mo ($t_{1/2} = 66$ hours) decays to ^{99m}Tc ($t_{1/2} = 6$ hours).

This is an important medical isotope generator used in nuclear medicine.

Activity ratio at equilibrium:

$$\frac{A_{Tc}}{A_{Mo}} = \frac{\lambda_{Tc}}{\lambda_{Tc} - \lambda_{Mo}} = \frac{0.693/6}{0.693/6 - 0.693/66} = \frac{11}{10} = 1.1$$

The technetium activity is 10% higher than the molybdenum activity.

3.4.4 No Equilibrium

If the daughter has a longer half-life than the parent ($t^B_{1/2} > t^A_{1/2}$), no equilibrium is established. The daughter activity continuously increases until the parent is depleted.

3.5 Artificial Radioactivity and Transmutation

3.5.1 Discovery of Artificial Radioactivity

In 1934, Irène Curie and Frédéric Joliot discovered that stable nuclei could be made radioactive by bombardment with alpha particles:

$$^{27}_{13}\text{Al} + ^{4}_{2}\text{He} \rightarrow ^{30}_{15}\text{P} + ^{1}_{0}\text{n} \tag{3.46}$$

The phosphorus-30 produced is radioactive:

$$^{30}_{15}\text{P} \rightarrow ^{30}_{14}\text{Si} + e^+ + \nu_e \quad (t_{1/2} = 2.5 \text{ min}) \tag{3.47}$$

This was the first artificial production of radioactive isotopes.

3.5.2 Nuclear Transmutation

Definition 3.5.1. Nuclear transmutation is the conversion of one element into another through nuclear reactions.

Methods of Producing Artificial Radioisotopes

1. **Neutron Activation**

   Neutrons, being uncharged, easily penetrate nuclei:

   $$ {}^{A}_{Z}X + {}^{1}_{0}n \rightarrow {}^{A+1}_{Z} X^{*} \rightarrow {}^{A+1}_{Z} X + \gamma \tag{3.48} $$

   Example (production of medical isotopes):

   $$ {}^{59}_{27}\mathrm{Co} + n \rightarrow {}^{60}_{27}\mathrm{Co} + \gamma \tag{3.49} $$

2. **Charged Particle Bombardment**

   Using accelerators to bombard targets with protons, deuterons, or alpha particles:

   $$ {}^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} \tag{3.50} $$

3. **Photonuclear Reactions**

   High-energy gamma rays can induce nuclear reactions:

   $$ {}^{A}_{Z}X + \gamma \rightarrow {}^{A-1}_{Z} X + n \tag{3.51} $$

4. **Fission Products**

   Nuclear fission produces many radioactive isotopes:

   $$ {}^{235}_{92}\mathrm{U} + n \rightarrow \text{fission fragments (radioactive)} + \text{neutrons} \tag{3.52} $$

3.5.3 Nuclear Reactors and Isotope Production

Nuclear reactors are major sources of artificial radioisotopes. The high neutron flux in a reactor enables efficient neutron activation.

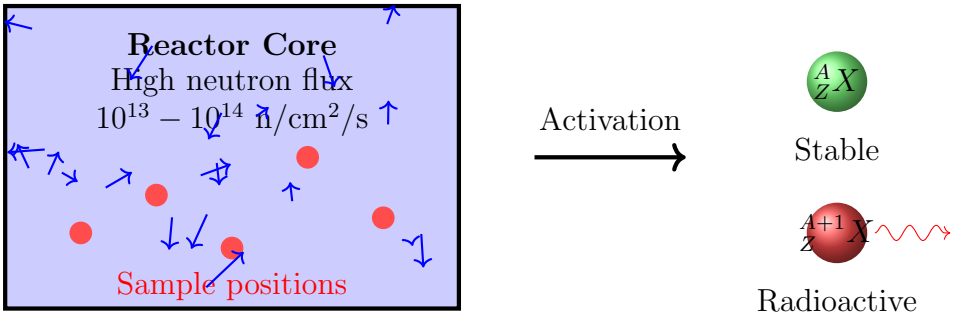


Figure 3.9: Neutron activation in a nuclear reactor

3.5.4 Particle Accelerators

Cyclotrons, linear accelerators, and other machines produce radioisotopes for medical and research applications, especially those that cannot be made in reactors.

Common Accelerator-Produced Isotopes

- ^{18}F: PET imaging ($t_{1/2} = 110$ min)
- ^{11}C: PET imaging ($t_{1/2} = 20$ min)
- ^{13}N: PET imaging ($t_{1/2} = 10$ min)
- ^{123}I: SPECT imaging ($t_{1/2} = 13$ hours)

3.6 Radioisotopes and Their Applications

Radioactive isotopes have become indispensable tools in medicine, industry, agriculture, and research. Their applications exploit various properties: penetrating radiation, traceable behavior, and controllable half-lives.

3.6.1 Medical Applications

1. Diagnostic Imaging

Nuclear Medicine Imaging

Radioisotopes are used as tracers to visualize organ function and metabolism:

- **Technetium-99m** ($t_{1/2} = 6$ hours): Most widely used medical isotope
  - Bone scans
  - Heart perfusion studies
  - Kidney function tests
- **Iodine-131** ($t_{1/2} = 8$ days): Thyroid imaging and treatment
- **Fluorine-18** ($t_{1/2} = 110$ min): PET scans for cancer detection
- **Thallium-201** ($t_{1/2} = 73$ hours): Cardiac imaging

PET Scanning

Positron Emission Tomography (PET) uses positron-emitting isotopes like ^{18}F. Process:

1. Radiotracer (e.g., ^{18}F-FDG) injected into patient
2. Isotope accumulates in metabolically active tissues (tumors)
3. Positron emission: $e^+ + e^- \rightarrow 2\gamma$ (511 keV, opposite directions)
4. Coincidence detection locates emission point
5. 3D image reconstructed showing metabolic activity

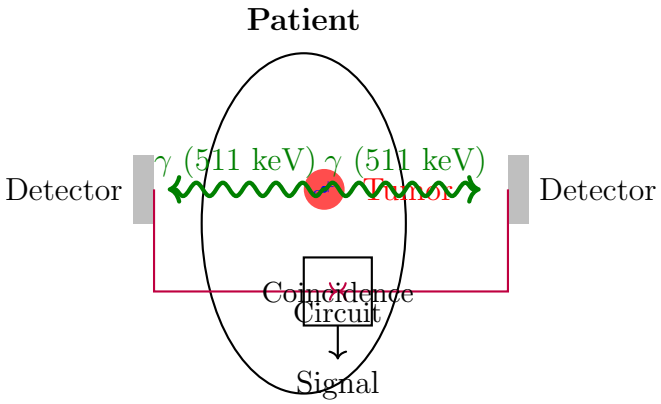


Figure 3.10: Principle of PET scanning using positron-emitting radioisotopes

2. Radiation Therapy

Radioisotopes are used to destroy cancer cells:

- **External Beam Therapy**:
  - ^{60}Co teletherapy units (gamma rays: 1.17, 1.33 MeV)
  - Linear accelerators (X-rays, electrons)
- **Brachytherapy** (internal radiation):
  - ^{192}Ir: Temporary implants
  - ^{125}I: Permanent prostate seed implants
  - ^{137}Cs: Gynecological cancers
- **Systemic Radiotherapy**:
  - ^{131}I: Thyroid cancer treatment
  - ^{89}Sr: Bone metastases pain relief
  - ^{90}Y: Liver cancer microspheres

3. Sterilization

^{60}Co gamma rays sterilize medical equipment, pharmaceuticals, and disposable medical supplies without heat or chemicals.

3.6.2 Industrial Applications

1. Radiography

Industrial radiography inspects welds, castings, and structures:

- ^{192}Ir ($t_{1/2} = 74$ days): Common for field radiography
- ^{60}Co: Thick steel sections
- ^{137}Cs: Medium-density materials

2. Thickness and Level Gauging

Beta or gamma sources measure material thickness in manufacturing:

$$I = I_0 e^{-\mu x} \tag{3.53}$$

where μ is the attenuation coefficient and x is the thickness.
Applications:

- Paper manufacturing (thickness control)
- Steel rolling (thickness measurement)
- Fill level in tanks and silos

3. Tracers in Industry

Radioactive tracers study:

- Pipeline flows and leaks
- Mixing efficiency in chemical plants
- Wear in machinery (radioactive oil additives)

3.6.3 Environmental and Agricultural Applications

1. Radiocarbon Dating

Carbon-14 dating determines the age of organic materials up to ∼50,000 years old.
Principle:

- Cosmic rays produce ^{14}C in the atmosphere
- Living organisms maintain constant ^{14}C/^{12}C ratio
- After death, ^{14}C decays ($t_{1/2} = 5730$ years)

- Measuring remaining ^{14}C gives age

$$t = \frac{1}{\lambda} \ln \frac{A_0}{A(t)} = \frac{t_{1/2}}{0.693} \ln \frac{A_0}{A(t)} \tag{3.54}$$

Example 3.6.1. Carbon Dating

A wooden artifact has ^{14}C activity of 8.0 decays/min per gram of carbon. Living wood has 15.3 decays/min per gram. How old is the artifact?

Solution:

$$\begin{aligned} t &= \frac{5730}{0.693} \ln \frac{15.3}{8.0} \\ &= 8270 \ln(1.91) \\ &= 8270 \times 0.647 \\ &= 5350 \text{ years} \end{aligned}$$

The artifact is approximately 5,350 years old.

2. Agricultural Applications

- **Insect Control**: Sterilizing male insects with radiation for pest control
- **Food Preservation**: Gamma irradiation extends shelf life
- **Crop Improvement**: Mutation breeding using radiation
- **Fertilizer Studies**: Radioactive tracers (^{32}P, ^{15}N) optimize fertilizer use

3. Hydrology

Radioactive tracers study:

- Groundwater movement
- River flow rates
- Sediment transport
- Water pollution dispersion

3.6.4 Research Applications

1. Neutron Activation Analysis (NAA)

Highly sensitive elemental analysis technique:

1. Sample irradiated with neutrons
2. Elements become radioactive with characteristic half-lives
3. Gamma ray spectroscopy identifies and quantifies elements
4. Detection limits: parts per billion or better

Applications: forensics, archaeology, environmental monitoring, quality control

2. Autoradiography

Visualizes distribution of radioactive substances in biological samples, used extensively in molecular biology and biochemistry.

3. Radioisotope Tracers in Research

- Metabolism studies (^{14}C, ^{3}H)
- Drug development and pharmacokinetics
- Protein synthesis (^{35}S, ^{3}H)
- DNA/RNA studies (^{32}P)

Isotope	Half-life	Decay	Main Applications
^{3}H (Tritium)	12.3 y	β^-	Biological tracers, luminous paint
^{14}C	5730 y	β^-	Radiocarbon dating, tracers
^{32}P	14.3 d	β^-	Molecular biology, cancer therapy
^{60}Co	5.27 y	β^-, γ	Radiotherapy, sterilization
^{99m}Tc	6.0 h	γ	Medical imaging (SPECT)
^{131}I	8.0 d	β^-, γ	Thyroid treatment, imaging
^{137}Cs	30.2 y	β^-, γ	Radiation source, gauges
^{192}Ir	73.8 d	β^-, γ	Industrial radiography
^{18}F	110 min	β^+	PET imaging
^{241}Am	432 y	α	Smoke detectors

Table 3.2: Common radioisotopes and their applications

3.6.5 Safety Considerations

Working with radioisotopes requires careful safety protocols:

Radiation Safety Principles: ALARA

ALARA: As Low As Reasonably Achievable
Three main protection methods:

1. **Time**: Minimize exposure duration
2. **Distance**: Maximize distance from source (intensity $\propto 1/r^2$)
3. **Shielding**: Use appropriate barriers (lead, concrete, water)

Additional measures:

- Personal dosimeters to monitor exposure
- Proper handling procedures and training
- Containment and waste disposal protocols
- Regular monitoring and surveys

3.7 Summary

In this chapter, we have explored the fundamental aspects of radioactivity and its wide-ranging applications:

1. **Types of Radioactive Decay**:
   - Alpha decay: emission of ${}^4_2\mathrm{He}$ nuclei, occurs in heavy nuclei
   - Beta decay: β^- (electron), β^+ (positron), or electron capture
   - Gamma decay: emission of high-energy photons from excited nuclear states
2. **Radioactive Decay Law**:
   - Exponential decay: $N(t) = N_0 e^{-\lambda t}$
   - Activity: $A(t) = \lambda N(t) = A_0 e^{-\lambda t}$
3. **Key Parameters**:
   - Decay constant: λ (s^{-1})
   - Half-life: $t_{1/2} = 0.693/\lambda$
   - Mean life: $\tau = 1/\lambda = 1.443\, t_{1/2}$
4. **Successive Decay**:
   - Secular equilibrium: parent half-life $\gg$ daughter half-life
   - Transient equilibrium: parent half-life $>$ daughter half-life
5. **Artificial Radioactivity**:
   - Neutron activation in reactors
   - Charged particle bombardment in accelerators
   - Fission products
6. **Applications**:
   - Medical: diagnosis (PET, SPECT), therapy (cancer treatment), sterilization
   - Industrial: radiography, gauging, tracers
   - Environmental: radiocarbon dating, hydrology
   - Research: NAA, autoradiography, tracers

Understanding radioactivity is essential for harnessing the beneficial applications of nuclear radiation while maintaining appropriate safety standards. The principles developed in this chapter underpin much of modern nuclear technology and continue to drive innovations in medicine, industry, and scientific research.

3.8 Problems

1. The half-life of ^{222}Rn is 3.82 days.
   (a) Calculate the decay constant in s^{-1}.
   (b) If you start with 10^{20} atoms, how many remain after 10 days?
   (c) What is the initial activity in Bq and Ci?

2. A radioactive sample has an initial activity of 8.0 mCi. After 12 hours, the activity is 2.0 mCi. Calculate:
   (a) The decay constant
   (b) The half-life
   (c) The mean life

3. Show that after n half-lives, the fraction of radioactive nuclei remaining is $(1/2)^n$.

4. A sample of ^{131}I ($t_{1/2} = 8.0$ days) has an activity of 100 μCi. How long will it take for the activity to decrease to 10 μCi?

5. Calculate the specific activity (in Ci/g) of pure ^{60}Co ($t_{1/2} = 5.27$ years, $M = 60$ g/mol).

6. For the decay chain $A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C$ with $\lambda_A = 0.1$ h^{-1} and $\lambda_B = 0.05$ h^{-1}:
   (a) Is equilibrium possible?
   (b) If so, what type of equilibrium is established?
   (c) Calculate the activity ratio A_B/A_A at equilibrium.

7. A sample of wood from an archaeological site has a ^{14}C activity of 4.8 disintegrations per minute per gram of carbon. Living wood has an activity of 15.3 dpm/g. The half-life of ^{14}C is 5730 years. Determine the age of the sample.

8. Write the complete nuclear equations for:
   (a) Alpha decay of ^{238}U
   (b) Beta-minus decay of ^{90}Sr
   (c) Beta-plus decay of ^{22}Na
   (d) Electron capture by ^{125}I

9. A ^{99}Mo/^{99m}Tc generator contains 500 mCi of ^{99}Mo ($t_{1/2} = 66$ h) at time $t = 0$.
   (a) Calculate the maximum activity of ^{99m}Tc ($t_{1/2} = 6$ h) and when it occurs.
   (b) What is the activity ratio at transient equilibrium?

10. The alpha particles from ^{238}U decay have energy 4.2 MeV. Calculate:
    (a) The Q-value of the decay
    (b) The recoil energy of the daughter nucleus (^{234}Th)

11. A hospital receives a shipment of ^{18}F ($t_{1/2}$ = 110 min) with activity 200 mCi at 8:00 AM. What will be the activity at 2:00 PM?

12. Show that for secular equilibrium, the total activity is approximately twice the parent activity.

13. Calculate the number of atoms in a 1 mCi source of ^{137}Cs ($t_{1/2}$ = 30.2 years).

3.9 Further Reading

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Chapter 4

Nuclear Forces

4.1 Introduction

The existence of atomic nuclei poses one of the most fundamental questions in physics: what force is strong enough to bind protons and neutrons together in the incredibly small volume of the nucleus, overcoming the enormous electromagnetic repulsion between positively charged protons? The answer lies in the **nuclear force**, also known as the strong force, which is one of the four fundamental forces of nature.

The nuclear force is unlike any force encountered in classical physics or atomic physics. It operates at extremely short ranges (on the order of femtometers), is incredibly strong at these distances, and exhibits unique properties that distinguish it from electromagnetic and gravitational interactions. Understanding the nuclear force is essential for comprehending nuclear structure, stability, reactions, and the processes that power stars.

Historically, the study of nuclear forces began with Rutherford's discovery of the nucleus in 1911. However, it was not until the 1930s that Hideki Yukawa proposed the first theoretical framework for understanding nuclear forces through his meson exchange theory. This revolutionary idea—that forces are mediated by the exchange of particles—laid the foundation for modern particle physics and quantum field theory.

In this chapter, we will explore:

- The fundamental characteristics that define nuclear forces
- Yukawa's meson exchange theory and the concept of force-mediating particles
- The Yukawa potential and its physical implications
- Key properties: charge independence and saturation
- Experimental evidence from nucleon-nucleon scattering
- The role of spin in nuclear interactions

Understanding nuclear forces is not merely of theoretical interest—it underpins our comprehension of nuclear stability, radioactive decay, nuclear reactions, and fusion processes that power both stars and potential future energy sources.

4.2 Characteristics of Nuclear Forces

Through decades of experimental observations and theoretical developments, physicists have identified several distinctive characteristics of the nuclear force that set it apart from other fundamental interactions.

4.2.1 Short Range

The most striking characteristic of the nuclear force is its extremely short range.

Range of Nuclear Force

The nuclear force is effectively zero beyond a distance of approximately 2-3 femtometers (fm), where 1 fm = 10^{-15} m.
Key observations:

- Strongly attractive at distances of 1-2 fm
- Becomes repulsive at distances less than ∼0.5 fm (hard core)
- Negligible beyond ∼3 fm

This short-range nature explains why nuclear forces don't affect atomic or molecular physics, and why nuclei don't grow indefinitely—only neighboring nucleons interact significantly.

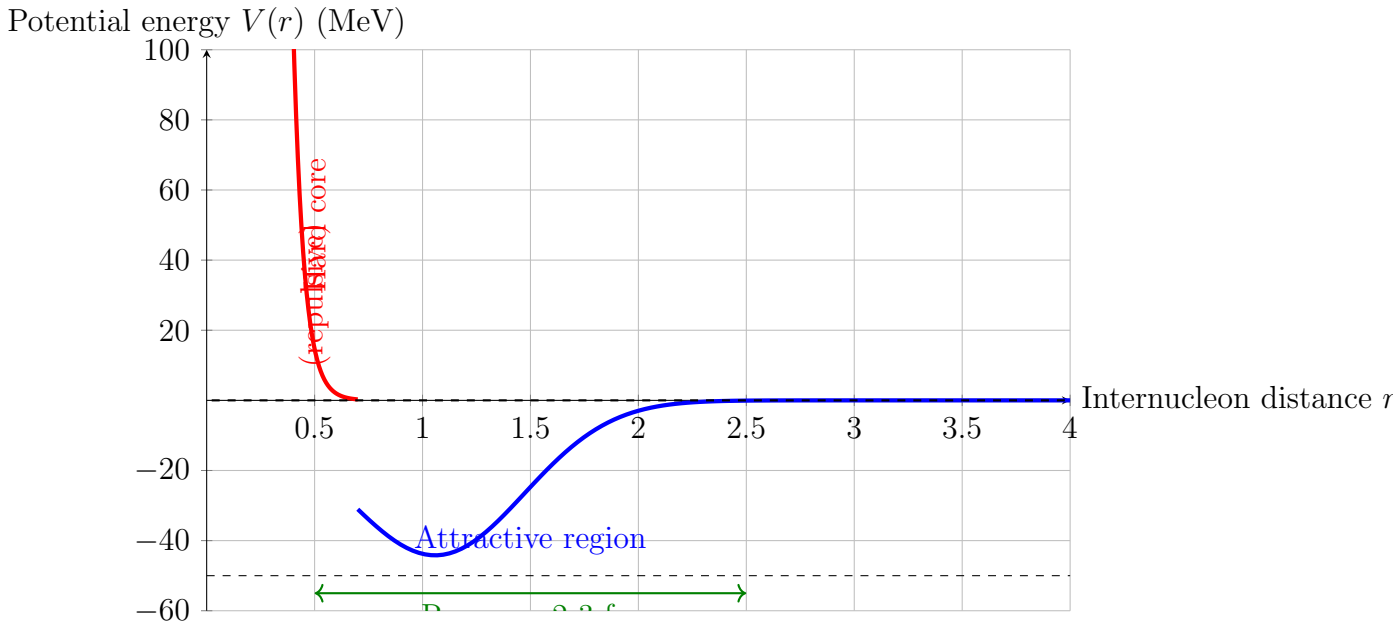

Figure 4.1: Schematic representation of the nucleon-nucleon potential showing the repulsive core, attractive well, and short range

4.2.2 Strong Attraction

At typical nuclear distances (around 1-2 fm), the nuclear force is strongly attractive.

- The depth of the potential well is approximately 40-50 MeV
- This is much stronger than electromagnetic forces at nuclear scales
- It must overcome the Coulomb repulsion between protons ($\sim$ few MeV)

To quantify the strength, consider two protons separated by 1 fm:
Coulomb repulsion energy:

$$E_{\text{Coulomb}} = \frac{ke^2}{r} = \frac{(1.44 \text{ MeV·fm})}{1 \text{ fm}} \approx 1.44 \text{ MeV} \tag{4.1}$$

Nuclear binding energy: ~ 8 MeV per nucleon (from binding energy curve)
The nuclear force must be much stronger than the Coulomb force to provide net binding.

4.2.3 Repulsive Core

At very short distances (less than about 0.5 fm), the nuclear force becomes strongly repulsive. This "hard core" prevents nucleons from collapsing into each other and maintains the finite size of nuclei.

Physical Origin of the Hard Core

The repulsive core arises from:

1. The Pauli exclusion principle (quantum mechanical effect)
2. The quark structure of nucleons (at very short distances, quarks begin to overlap)
3. Vector meson exchange (particularly ω and ρ mesons)

This hard core is essential for nuclear stability and explains why nuclear matter has approximately constant density.

4.2.4 Charge Independence

One of the most remarkable properties of the nuclear force is its approximate independence of electric charge.

Definition 4.2.1. Charge independence means that the strong nuclear force between two nucleons is approximately the same whether they are:

- Two protons (p-p)
- Two neutrons (n-n)
- A proton and a neutron (p-n)

More precisely, when electromagnetic effects are removed:

$$V_{pp} \approx V_{nn} \approx V_{pn} \tag{4.2}$$

Experimental Evidence:

1. **Mirror Nuclei:** Pairs of nuclei with Z and N interchanged have very similar binding energies after correcting for Coulomb energy.

   Example: ${}^{15}_{7}\mathrm{N}$ and ${}^{15}_{8}\mathrm{O}$

2. **Scattering Experiments:** Proton-proton (p-p) and neutron-proton (n-p) scattering show similar interaction strengths after Coulomb corrections.

This charge independence suggests that the nuclear force is a manifestation of a more fundamental interaction that doesn't distinguish between protons and neutrons—the strong interaction between quarks.

4.2.5 Saturation

The nuclear force exhibits a property called **saturation**, meaning each nucleon interacts strongly only with its nearest neighbors, not with all other nucleons in the nucleus.

Evidence for Saturation

1. **Constant Density:** Nuclear radius $R = R_0 A^{1/3}$ implies constant density, suggesting nucleons are closely packed like molecules in a liquid drop.

2. **Binding Energy per Nucleon:** BE/A is approximately constant ($\sim$8 MeV) for medium and heavy nuclei, not proportional to A.

   If each nucleon interacted with all others:

$$BE \propto \frac{A(A-1)}{2} \approx \frac{A^2}{2} \tag{4.3}$$

   But we observe:

$$BE \propto A \quad \text{(for large } A\text{)} \tag{4.4}$$

Saturation arises from:

- The short range of the force (only nearest neighbors interact)
- The repulsive core (prevents unlimited compression)
- The exchange character of the force (discussed in the next section)

4.2.6 Non-Central Forces and Tensor Components

Unlike the electromagnetic force (which is purely central, depending only on distance), the nuclear force has both central and non-central components.

The nuclear force depends on:

- The distance between nucleons: $\vec{r}$
- The spins of the nucleons: $\vec{s}_1, \vec{s}_2$
- The orbital angular momentum: $\vec{L}$

The general form includes:

$$V(\vec{r}) = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\vec{L}\cdot\vec{S} + \dots \tag{4.5}$$

where:

- $V_C(r)$: central potential
- $V_T(r)S_{12}$: tensor force term, $S_{12} = 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$
- $V_{LS}(r)\vec{L}\cdot\vec{S}$: spin-orbit coupling

The tensor force is particularly important:

- Explains the non-spherical charge distribution in the deuteron
- Contributes to the binding of the deuteron
- Affects excited nuclear states

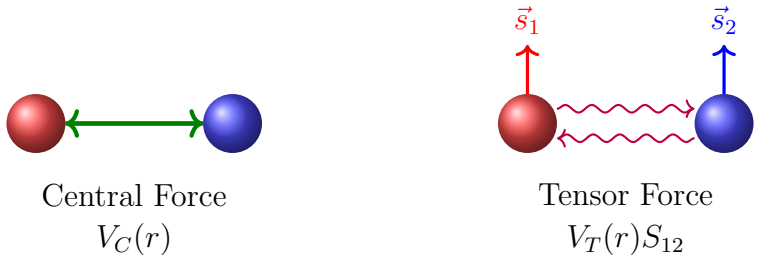

Figure 4.2: Comparison of central and tensor components of nuclear force

4.2.7 Spin Dependence

The nuclear force strongly depends on the relative spin orientation of the nucleons.

Key Observation: The deuteron (one proton + one neutron) has:

- A bound state with spins parallel (triplet state, $S = 1$, binding energy 2.224 MeV)
- No bound state with spins antiparallel (singlet state, $S = 0$)

This demonstrates that the nuclear force is more attractive when spins are aligned. The spin-dependent part can be written as:

$$V_{\text{spin}}(r) = V_s(r)\,\vec{\sigma}_1 \cdot \vec{\sigma}_2 \tag{4.6}$$

where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are Pauli spin matrices for the two nucleons.

4.3 Exchange Forces and Meson Theory

4.3.1 The Concept of Exchange Forces

In classical physics, forces are described by fields (like the electromagnetic field). However, in quantum field theory, forces arise from the exchange of particles called **mediators** or **force carriers**.

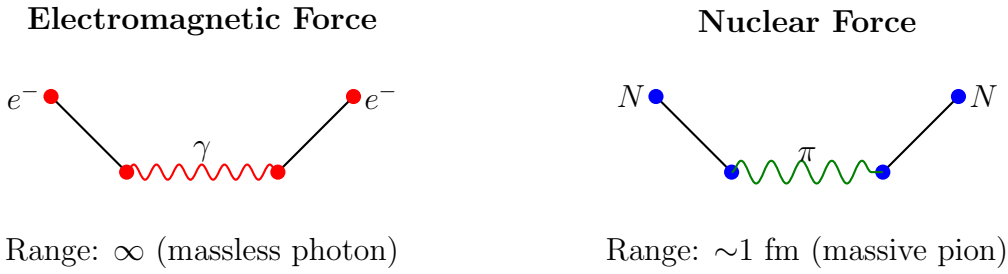

Figure 4.3: Comparison of electromagnetic force (photon exchange) and nuclear force (meson exchange)

Key Principle: The range of a force is inversely related to the mass of the exchange particle.

4.3.2 Yukawa's Meson Theory (1935)

Hideki Yukawa made a revolutionary proposal: the nuclear force is mediated by the exchange of particles with mass intermediate between electrons and protons—which he called **mesons**.

Yukawa's Reasoning

Derivation of Yukawa Potential

Starting from relativistic quantum mechanics, the wave equation for a particle with mass m is the Klein-Gordon equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = \frac{m^2c^2}{\hbar^2}\phi \tag{4.7}$$

For a static field ($\partial/\partial t = 0$):

$$\nabla^2\phi = \frac{m^2c^2}{\hbar^2}\phi \tag{4.8}$$

Let $\mu = mc/\hbar$. Then:

$$\nabla^2\phi = \mu^2\phi \tag{4.9}$$

For a point source at the origin (analogous to a point charge for the Coulomb potential):

$$\nabla^2\phi - \mu^2\phi = -g\delta^3(\vec{r}) \tag{4.10}$$

where g is the coupling constant (analogous to charge).
The solution for spherical symmetry is:

$$\phi(r) = -\frac{g}{4\pi r}e^{-\mu r} \tag{4.11}$$

The corresponding potential energy is:

$$\boxed{V(r) = -g^2\frac{e^{-\mu r}}{r} = -V_0\frac{e^{-r/a}}{r/a}} \tag{4.12}$$

where $a = \hbar/(mc) = 1/\mu$ is the range of the force.

This is the **Yukawa potential**, characterized by:

- Exponential decay with range $a = \hbar/(mc)$
- Reduces to Coulomb potential when $m \to 0$
- Short range for massive exchange particles

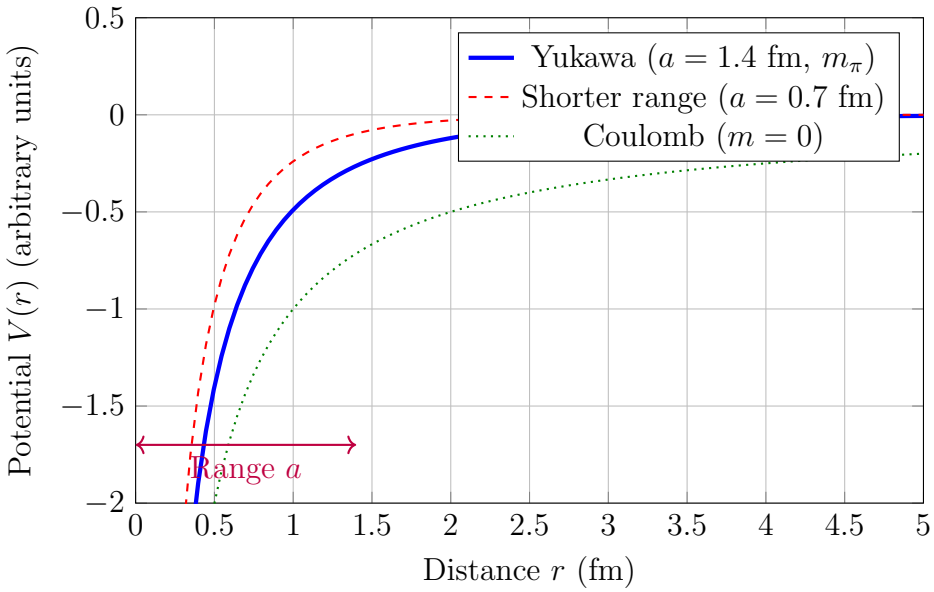

Figure 4.4: Yukawa potential compared with Coulomb potential and shorter-range variant

Prediction of Meson Mass

Yukawa used the known range of nuclear forces to predict the mass of the exchange particle.

Given:

- Range of nuclear force: $a \approx 1.4$ fm $= 1.4 \times 10^{-15}$ m
- Relation: $a = \hbar/(mc)$

Solving for mass:

$$m = \frac{\hbar}{ac} \tag{4.13}$$

$$= \frac{1.055 \times 10^{-34} \text{ J·s}}{(1.4 \times 10^{-15} \text{ m})(3 \times 10^{8} \text{ m/s})} \tag{4.14}$$

$$= 2.5 \times 10^{-28} \text{ kg} \tag{4.15}$$

In energy units:

$$mc^2 = \frac{\hbar c}{a} = \frac{197.3 \text{ MeV·fm}}{1.4 \text{ fm}} \approx 140 \text{ MeV} \tag{4.16}$$

Yukawa's Prediction

Yukawa predicted the existence of a particle with mass approximately 140 MeV/c^2, about 270 times the electron mass.
This particle was later discovered to be the **pion** (π meson):

- π^+, π^-, π^0 (charged and neutral pions)
- Mass: $m_{\pi^\pm} \approx 140$ MeV/c^2, $m_{\pi^0} \approx 135$ MeV/c^2
- Discovered in cosmic rays (1947) by Powell, Occhialini, and Lattes

Yukawa received the Nobel Prize in Physics in 1949 for this prediction.

4.3.3 Pion Exchange Mechanism

The nuclear force arises from the exchange of virtual pions between nucleons.

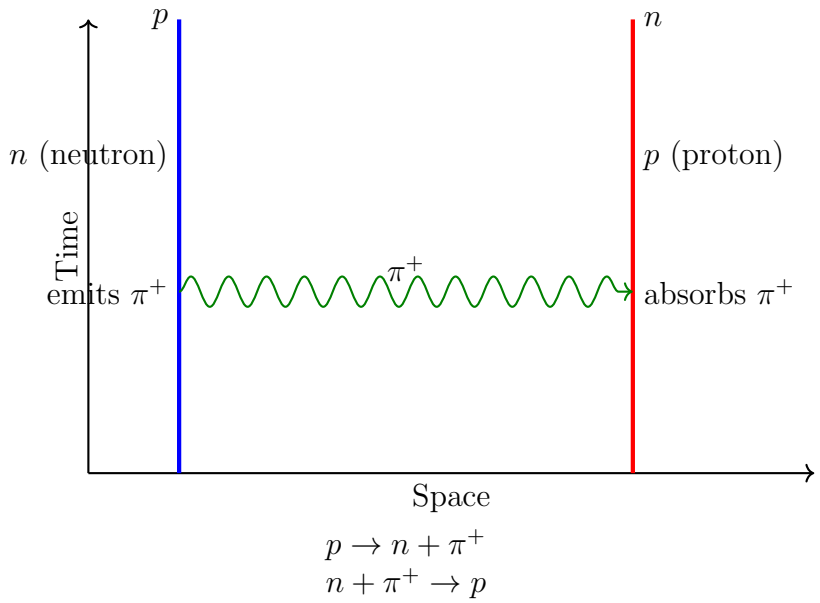

Figure 4.5: Feynman diagram showing pion exchange between nucleons

The exchange process:

1. A nucleon emits a virtual pion
2. The pion travels to the other nucleon
3. The second nucleon absorbs the pion
4. This exchange creates an attractive force

Energy-time uncertainty:
The emission and absorption of the pion violates energy conservation by $\Delta E \approx m_\pi c^2 = 140$ MeV. However, the Heisenberg uncertainty principle allows this for a short time:

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2} \tag{4.17}$$

The maximum time available is:

$$\Delta t \sim \frac{\hbar}{m_\pi c^2} \tag{4.18}$$

The maximum distance the pion can travel is:

$$r_{\max} = c \cdot \Delta t \sim \frac{\hbar c}{m_\pi c^2} = \frac{\hbar}{m_\pi c} \approx 1.4 \text{ fm} \tag{4.19}$$

This confirms the range of the nuclear force!

4.3.4 Modern Understanding: Multiple Meson Exchange

While the pion plays a dominant role, the nuclear force actually involves exchange of multiple mesons:

Meson	**Mass (MeV/c^2)**	**Range (fm)**	**Role**
π (pion)	140	$\sim$1.4	Long-range attraction
ρ (rho)	770	$\sim$0.25	Medium-range
ω (omega)	782	$\sim$0.25	Repulsive core

Table 4.1: Mesons involved in nuclear force

The complete nucleon-nucleon potential includes:

$$V(r) = V_\pi(r) + V_\rho(r) + V_\omega(r) + \ldots \tag{4.20}$$

Modern perspective: At a deeper level, the nuclear force is understood as a residual effect of the strong interaction between quarks (mediated by gluons), similar to how molecular forces are residual electromagnetic forces.

4.4 Charge Independence and Isotopic Spin

4.4.1 Experimental Evidence for Charge Independence

As mentioned earlier, the nuclear force is approximately the same for p-p, n-n, and p-n interactions (after removing electromagnetic effects).

Evidence from binding energies:

Consider isobaric triplets (same A, different Z):

Nucleus	Z	N	**Binding Energy (MeV)**
^{14}C	6	8	105.29
^{14}N	7	7	104.66
^{14}O	8	6	98.73

Table 4.2: Binding energies of isobaric triplet (after Coulomb correction, values are similar)

When Coulomb energy differences are subtracted, the binding energies are nearly equal, suggesting the nuclear force doesn't distinguish between protons and neutrons.

4.4.2 Isotopic Spin (Isospin)

To formalize charge independence, physicists introduced the concept of **isotopic spin** or **isospin**.

Definition 4.4.1. Isospin is a quantum number analogous to ordinary spin, but in an abstract "charge space" rather than real space.

For nucleons:

- Isospin quantum number: $I = 1/2$ (like spin $s = 1/2$)
- Third component I_3 (like spin projection m_s):
  - Proton: $I_3 = +1/2$
  - Neutron: $I_3 = -1/2$

In this formalism, proton and neutron are two states of the same particle—the **nucleon**—distinguished only by their isospin projection.

Charge independence means the nuclear force is invariant under isospin rotations:

$$V_{NN}(\vec{r}) \text{ is independent of } I_3 \tag{4.21}$$

For a nucleus with A nucleons:

- Total isospin: $\vec{I} = \sum_{i=1}^{A} \vec{I}_i$
- Magnitude: $I = 0, 1/2, 1, 3/2, \ldots$
- Third component: $I_3 = \sum_{i=1}^{A} I_{3,i} = \frac{Z-N}{2}$

Isospin Multiplets

Nuclei with the same A and I but different I_3 form an **isospin multiplet**.
Example: $A = 14$, $I = 1$ triplet:

- ^{14}C: $I_3 = -1$ (6p, 8n)
- ^{14}N: $I_3 = 0$ (7p, 7n)
- ^{14}O: $I_3 = +1$ (8p, 6n)

These should have nearly identical nuclear properties (energy levels, spins, parities) when electromagnetic effects are removed.

4.5 Saturation Properties of Nuclear Forces

The saturation of nuclear forces is crucial for understanding nuclear structure and stability. Let's examine this property in detail.

4.5.1 Manifestations of Saturation

1. Constant Density:

Nuclear volume is proportional to A:

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi (R_0 A^{1/3})^3 \propto A \tag{4.22}$$

Density:

$$\rho = \frac{A \cdot m_u}{V} = \frac{A \cdot m_u}{\frac{4}{3}\pi R_0^3 A} = \frac{3m_u}{4\pi R_0^3} = \text{constant} \tag{4.23}$$

2. Linear dependence of binding energy:

For large A, the binding energy is approximately:

$$BE \approx a_V A - a_S A^{2/3} - \ldots \approx \text{constant} \times A \tag{4.24}$$

This implies $BE/A \approx$ constant for large nuclei.

4.5.2 Causes of Saturation

1. Short Range:

Only nucleons within about 2-3 fm interact significantly. In the nuclear interior, each nucleon interacts with only a few neighbors (typically 8-12), not with all A nucleons.

2. Exchange Character:

The Pauli exclusion principle limits the number of nucleons that can occupy nearby quantum states and interact via exchange forces.

3. Repulsive Core:

The hard core prevents unlimited compression, maintaining a minimum separation between nucleons.

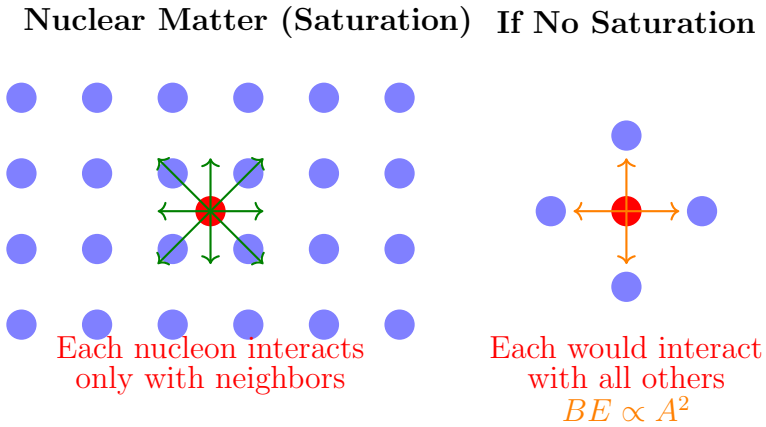
Nuclear Matter (Saturation)
If No Saturation
Each nucleon interacts only with neighbors
Each would interact with all others
$BE \propto A^2$

Figure 4.6: Saturation of nuclear forces: each nucleon interacts only with nearest neighbors

4.5.3 Consequences of Saturation

1. **Nuclear matter behaves like a liquid:**
   - Nearly incompressible (constant density)

- Approximately constant binding energy per particle
- Surface tension effects (surface energy term in SEMF)

2. **Large nuclei are unstable:**

   As A increases:

   - Nuclear binding grows as $\sim A$
   - Coulomb repulsion grows as $\sim Z^2 \sim A^2$
   - Eventually Coulomb wins, limiting nuclear size

3. **Nuclear density is universal:**

   All nuclei have approximately the same central density:

$$\rho_0 \approx 0.17 \text{ nucleons/fm}^3 \approx 2.3 \times 10^{17} \text{ kg/m}^3 \tag{4.25}$$

4.6 Nucleon-Nucleon Scattering

Scattering experiments provide the most direct information about nuclear forces. By colliding nucleons and measuring the scattering cross-sections and angular distributions, we can probe the interaction potential.

4.6.1 Types of Nucleon-Nucleon Scattering

1. **Proton-Proton (p-p) Scattering:**

   - Includes both nuclear and Coulomb interactions
   - Easiest experimentally (proton beams and targets readily available)
   - Coulomb effects must be carefully subtracted

2. **Neutron-Proton (n-p) Scattering:**

   - Pure nuclear interaction (no Coulomb)
   - Most direct probe of nuclear force
   - Experimentally more challenging (neutron beams required)

3. **Neutron-Neutron (n-n) Scattering:**

   - Extremely difficult (no free neutron targets)
   - Inferred from nuclear reactions or special techniques

4.6.2 Scattering Cross-Section

Definition 4.6.1. The **differential cross-section** $d\sigma/d\Omega$ describes the probability of scattering into a solid angle $d\Omega$ at angle θ:

$$\frac{d\sigma}{d\Omega}(\theta) = \text{number of particles scattered into } d\Omega \text{ per unit time} \tag{4.26}$$

The **total cross-section** is:

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega \tag{4.27}$$

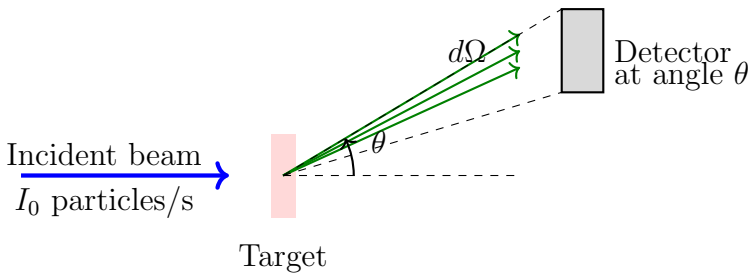

Figure 4.7: Schematic of a scattering experiment

4.6.3 Energy Dependence

The scattering cross-section depends strongly on energy:

- **Low energy (< 10 MeV):** Dominated by s-wave scattering (spherically symmetric)
- **Medium energy (10-300 MeV):** Higher partial waves contribute, revealing details of potential
- **High energy (> 300 MeV):** Inelastic processes dominate (pion production, etc.)

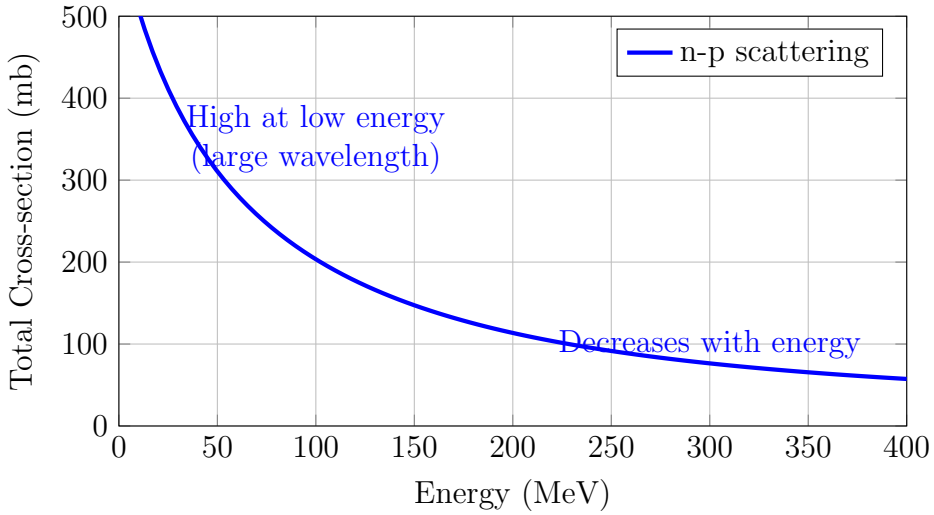

Figure 4.8: Qualitative behavior of n-p total cross-section vs. energy

4.6.4 Angular Distribution

The angular distribution reveals the nature of the force:

- **Isotropic (constant with θ):** Suggests s-wave scattering (low energy)
- **Forward peaked:** High energy scattering or Coulomb scattering
- **Diffraction pattern:** Indicates hard-sphere-like scattering

4.6.5 Spin Dependence in Scattering

The nuclear force depends on the relative spin states of the colliding nucleons.

Singlet and Triplet States

For two spin-1/2 particles:

- **Singlet state** ($S = 0$): Spins antiparallel, $\uparrow\downarrow - \downarrow\uparrow$
- **Triplet state** ($S = 1$): Spins parallel, $\uparrow\uparrow$, or symmetric combinations

The interaction potential differs:

$$V_{singlet}(r) \neq V_{triplet}(r) \tag{4.28}$$

Evidence from deuteron:

- Triplet state ($S = 1$): Bound with $BE = 2.224$ MeV
- Singlet state ($S = 0$): No bound state (unbound)

This clearly demonstrates that $V_{triplet}$ is more attractive than $V_{singlet}$.

Scattering Lengths

At very low energies, scattering is characterized by the **scattering length** a:

$$\sigma_{tot} \approx 4\pi a^2 \quad (\text{for } E \to 0) \tag{4.29}$$

Experimental values for n-p scattering:

$$a_{triplet} = +5.42 \text{ fm} \quad (S = 1) \tag{4.30}$$

$$a_{singlet} = -23.7 \text{ fm} \quad (S = 0) \tag{4.31}$$

The large negative singlet scattering length indicates a virtual bound state (almost bound).

4.6.6 Information from Scattering Experiments

Modern n-p scattering data at various energies has yielded:

1. **Shape of the potential:** Generally a repulsive core plus attractive well
2. **Strength:** Well depth $V_0 \approx 40 - 50$ MeV
3. **Range:** Approximately 2-3 fm
4. **Spin dependence:** Triplet state more attractive than singlet
5. **Tensor force:** Required to explain deuteron properties
6. **Non-central components:** Spin-orbit and tensor terms necessary

Key Scattering Results

Low-energy n-p scattering:

- Total cross-section at thermal energies: $\sigma \approx 20$ barns
- Large scattering lengths indicate strong interaction
- Spin-dependence clearly observed

Phase shift analysis:
By analyzing how the scattering phase shifts vary with energy and angular momentum, physicists have reconstructed detailed nucleon-nucleon potentials with high precision.
Modern potentials (like the Argonne v_{18} potential) fit thousands of scattering data points with excellent accuracy.

4.7 Summary

In this chapter, we have explored the fundamental nature of nuclear forces:

1. **Characteristics of Nuclear Forces:**
   - Short range (∼2-3 fm)
   - Strong attraction at nuclear distances
   - Repulsive core at $r < 0.5$ fm
   - Charge independence
   - Saturation property
   - Spin dependence
   - Non-central (tensor) components
2. **Yukawa Meson Theory:**
   - Forces mediated by particle exchange
   - Yukawa potential: $V(r) = -g^2 e^{-\mu r}/r$
   - Range related to meson mass: $a = \hbar/(m_\pi c)$
   - Successful prediction of pion mass (∼140 MeV)
   - Modern understanding includes multiple mesons
3. **Charge Independence:**
   - $V_{pp} \approx V_{nn} \approx V_{pn}$ (after Coulomb correction)
   - Formalized through isospin symmetry
   - Nucleons treated as two states of same particle
   - Isospin multiplets have similar properties
4. **Saturation:**

- Each nucleon interacts only with nearest neighbors
- Leads to constant nuclear density
- Binding energy proportional to A for large nuclei
- Explains liquid-drop behavior of nuclear matter

5. **Nucleon-Nucleon Scattering:**
   - Direct probe of nuclear force
   - Cross-sections reveal potential shape and strength
   - Clear evidence for spin dependence
   - Phase shift analysis yields detailed potentials

The nuclear force, while complex, can be understood as arising from quark-quark interactions mediated by gluons, with the nucleon-nucleon force being a residual effect analogous to Van der Waals forces in atomic physics. The phenomenological potentials developed from scattering data provide accurate descriptions for nuclear structure calculations.

Understanding nuclear forces is fundamental to all of nuclear physics—from nuclear structure and reactions to astrophysical processes and applications in energy and medicine.

4.8 Problems

1. Using the Yukawa potential $V(r) = -g^2 e^{-\mu r}/r$:
   (a) Calculate the range parameter μ for the pion mass $m_\pi = 140$ MeV/c^2.
   (b) At what distance is the potential reduced to $1/e$ of its value at $r = 0$?

2. The deuteron has only one bound state with binding energy 2.224 MeV. Estimate the depth of the n-p potential well assuming a square well with range 2 fm.

3. Show that the Yukawa potential reduces to the Coulomb potential $V(r) = -g^2/r$ in the limit $\mu \to 0$ (massless exchange particle).

4. Calculate the electrostatic repulsion energy between two protons separated by 1.5 fm. Compare this to the typical nuclear binding energy per nucleon.

5. For an isospin triplet with $A = 14$:
   (a) Identify the three members (nuclei).
   (b) Calculate I_3 for each member.
   (c) Explain why their binding energies differ despite charge independence.

6. A beam of thermal neutrons (energy 0.025 eV) scatters from protons with a total cross-section of 20 barns.
   (a) Calculate the scattering length.
   (b) What would be the cross-section at 1 eV if it scales with wavelength?

7. If the nuclear force had infinite range (like electromagnetism), how would the binding energy scale with mass number A? Explain why this doesn't match observation.

8. Explain why the existence of stable nuclei with $A > 200$ would be impossible without the saturation property of nuclear forces.

9. The ω meson (mass 782 MeV/c^2) contributes to the repulsive core. Calculate the range of the force mediated by ω exchange.

10. Compare the ranges of forces mediated by:

    (a) Photon (electromagnetic force)

    (b) Pion (nuclear force)

    (c) W boson, mass 80 GeV/c^2 (weak force)

4.9 Further Reading

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Chapter 5

Nuclear Models

5.1 Introduction to Nuclear Models

5.1.1 The Challenge of Nuclear Structure

The atomic nucleus is one of the most fascinating and complex systems in physics. With a radius of approximately 10^{-15} meters (1 femtometer), the nucleus contains nearly all the mass of an atom within an incredibly small volume. Understanding the structure and behavior of nuclei requires sophisticated theoretical frameworks that can account for:

- The strong nuclear force binding nucleons together
- The electromagnetic repulsion between protons
- Quantum mechanical effects governing nucleon behavior
- The collective behavior of many interacting particles
- The remarkable stability patterns observed in nature

Unlike the hydrogen atom, where the Schrödinger equation can be solved exactly, the nuclear many-body problem presents formidable mathematical challenges. A nucleus with mass number A consists of A strongly interacting particles, each influencing all others. The exact solution of such a system is practically impossible for all but the lightest nuclei.

5.1.2 Historical Development

The development of nuclear models proceeded through several stages:

1. **1930s**: George Gamow proposed the liquid drop analogy
2. **1935**: Carl Friedrich von Weizsäcker developed the semi-empirical mass formula
3. **1949**: Maria Goeppert Mayer and J. Hans D. Jensen independently proposed the shell model
4. **1952-53**: Aage Bohr and Ben Mottelson developed the collective model

Each model provides unique insights into nuclear structure, and together they form a comprehensive framework for understanding nuclear phenomena.

5.1.3 Classification of Nuclear Models

Nuclear models can be broadly classified into two categories:

1. Macroscopic Models:

- Treat the nucleus as a continuous medium
- Focus on bulk properties
- Example: Liquid Drop Model

2. Microscopic Models:

- Consider individual nucleons and quantum states
- Focus on discrete energy levels
- Example: Shell Model

The Collective Model represents a synthesis, incorporating both macroscopic and microscopic aspects.

5.2 The Liquid Drop Model

5.2.1 Physical Motivation and Analogies

The liquid drop model, first proposed by George Gamow in 1929 and developed extensively by Niels Bohr, treats the atomic nucleus as a drop of incompressible nuclear liquid. This seemingly simple analogy is justified by several remarkable similarities between atomic nuclei and liquid drops:

Constant Density

Nuclear matter exhibits approximately constant density:

$$\rho_{\text{nuclear}} \approx 2.3 \times 10^{17} \text{ kg/m}^3 \tag{5.1}$$

This is independent of the nucleus size, just as water maintains constant density whether in a droplet or an ocean. The nuclear radius follows the empirical relation:

$$R = r_0 A^{1/3} \tag{5.2}$$

where $r_0 \approx 1.2$ fm and A is the mass number. This implies that nuclear volume is proportional to A:

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi r_0^3 A \propto A \tag{5.3}$$

Since mass is also proportional to A, the density remains constant.

Saturation of Nuclear Forces

The strong nuclear force exhibits a saturation property: each nucleon interacts strongly only with its immediate neighbors (typically 8-12 nucleons in the interior). This is analogous to intermolecular forces in liquids, where each molecule interacts primarily with nearby molecules.

The binding energy per nucleon approaches a constant value ($\approx$ 8 MeV) for medium to heavy nuclei, indicating that adding more nucleons doesn't proportionally increase the binding per nucleon—a hallmark of force saturation.

Incompressibility

Nuclei resist compression, behaving like incompressible fluids. The nuclear compressibility modulus is:

$$K \approx 200 - 300 \text{ MeV} \tag{5.4}$$

This large value indicates that significant energy is required to compress nuclear matter, similar to liquids.

Surface Tension

Just as liquid drops minimize surface area due to surface tension, nuclei exhibit surface energy effects. Nucleons at the nuclear surface are less tightly bound than those in the interior because they have fewer neighboring nucleons with which to interact.

5.2.2 The Semi-Empirical Mass Formula (SEMF)

The most successful application of the liquid drop model is the Weizsäcker mass formula, which expresses nuclear binding energy as a sum of terms representing different physical effects.

General Form

The binding energy $B(A, Z)$ of a nucleus with mass number A and atomic number Z (number of protons) is given by:

Weizsäcker Semi-Empirical Mass Formula

$$B(A, Z) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A - 2Z)^2}{A} + \delta(A, Z) \tag{5.5}$$

Standard Parameter Values:

- $a_V = 15.75$ MeV (volume term)
- $a_S = 17.8$ MeV (surface term)
- $a_C = 0.711$ MeV (Coulomb term)
- $a_A = 23.7$ MeV (asymmetry term)
- Pairing term: $\delta(A, Z)$ defined below

5.2.3 Detailed Analysis of Each Term

Volume Term: $a_V A$

Physical Origin: This term represents the attractive contribution from the strong nuclear force. Since each nucleon interacts with approximately the same number of neighbors (saturation), the total binding energy is proportional to the number of nucleons.

Mathematical Derivation

If each nucleon contributes an average binding energy ϵ_0, and there are A nucleons:

$$B_{\text{volume}} = \epsilon_0 \cdot A = a_V A \tag{5.6}$$

The coefficient $a_V \approx 15.75$ MeV represents the binding energy per nucleon in the absence of other effects. This is positive and contributes to nuclear stability.

Physical Interpretation: This term dominates for large nuclei and would give a binding energy per nucleon of about 16 MeV if it were the only contribution—close to the observed maximum of about 8.8 MeV for iron-56, indicating that competing negative terms are significant.

Surface Term: $-a_S A^{2/3}$

Physical Origin: Nucleons at the nuclear surface have fewer neighbors than those in the interior, resulting in reduced binding. This is directly analogous to surface tension in liquid drops.

Mathematical Derivation

The number of surface nucleons is proportional to the surface area:

$$\text{Surface area} = 4\pi R^2 = 4\pi r_0^2 A^{2/3} \tag{5.7}$$

Each surface nucleon has reduced binding energy. The total reduction is:

$$B_{\text{surface}} = -a_S A^{2/3} \tag{5.8}$$

where $a_S \approx 17.8$ MeV. The negative sign indicates this reduces the total binding energy.

Size Dependence: For small nuclei, the surface-to-volume ratio is large, making this term relatively more important. As A increases, the ratio $A^{2/3}/A = A^{-1/3}$ decreases, so the surface effect becomes proportionally less significant.

Coulomb Term: $-a_C \frac{Z^2}{A^{1/3}}$

Physical Origin: Protons repel each other electromagnetically. This Coulomb repulsion reduces nuclear binding and limits the size of stable nuclei.

Mathematical Derivation

Consider the nucleus as a uniformly charged sphere of radius $R = r_0 A^{1/3}$ with total charge $Q = Ze$.
The electrostatic self-energy of a uniformly charged sphere is:

$$E_C = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R} = \frac{3}{5}\frac{(Ze)^2}{4\pi\epsilon_0 r_0 A^{1/3}} \tag{5.9}$$

This can be written as:

$$B_{\text{Coulomb}} = -a_C \frac{Z^2}{A^{1/3}} \tag{5.10}$$

where:

$$a_C = \frac{3e^2}{20\pi\epsilon_0 r_0} \tag{5.11}$$

Substituting $r_0 \approx 1.2$ fm and fundamental constants:

$$a_C \approx 0.711 \text{ MeV} \tag{5.12}$$

Implications: This term grows rapidly with Z, explaining why:

- Heavy nuclei require more neutrons than protons ($N > Z$) for stability
- There is an upper limit to nuclear stability around $Z \approx 82 - 92$
- Very heavy nuclei are susceptible to fission

Asymmetry Term: $-a_A \frac{(A-2Z)^2}{A}$

Physical Origin: This term, also called the symmetry term, accounts for the preference of nuclei to have approximately equal numbers of protons and neutrons. This preference arises from the Pauli exclusion principle.

Mathematical Derivation

Nucleons (protons and neutrons) are fermions obeying the Pauli exclusion principle. They occupy separate energy level systems.
Consider the nucleus as having two independent Fermi gases: one for protons and one for neutrons.
For a given volume, the Fermi energy (highest occupied energy level) is:

$$E_F \propto N^{2/3} \tag{5.13}$$

where N is the number of particles.
The total kinetic energy is:

$$E_{\text{kinetic}} = \frac{3}{5} N E_F \propto N^{5/3} \tag{5.14}$$

For the nucleus with Z protons and $N = A - Z$ neutrons:

$$E_{\text{total}} \propto Z^{5/3} + (A - Z)^{5/3} \tag{5.15}$$

This is minimized when $Z = A - Z = A/2$, i.e., when $N = Z$.
Expanding around $Z = A/2$:

$$E_{\text{total}} \approx \text{const} + \text{coeff} \times (Z - A/2)^2 = \text{const} + \text{coeff} \times \frac{(A - 2Z)^2}{4} \tag{5.16}$$

This excess energy reduces binding:

$$B_{\text{asymmetry}} = -a_A \frac{(A - 2Z)^2}{A} \tag{5.17}$$

where $a_A \approx 23.7$ MeV.

Consequences:

- Light stable nuclei have $N \approx Z$
- Heavy stable nuclei have $N > Z$ (to offset Coulomb repulsion)
- The line of stability curves in the N-Z plane

Pairing Term: $\delta(A, Z)$

Physical Origin: Nuclear stability exhibits a preference for even numbers of protons and neutrons. This pairing effect arises from residual nucleon-nucleon interactions beyond the mean field.

The pairing term is defined as:

$$\delta(A, Z) = \begin{cases} +a_P A^{-1/2} & \text{even-even nuclei (Z even, N even)} \\ 0 & \text{odd-A nuclei} \\ -a_P A^{-1/2} & \text{odd-odd nuclei (Z odd, N odd)} \end{cases} \tag{5.18}$$

where $a_P \approx 11.18$ MeV.
Observational Evidence:

- Of about 280 stable nuclei, approximately 160 are even-even
- Only 4 stable odd-odd nuclei exist: ^{2}H, ^{6}Li, ^{10}B, ^{14}N

- Even-even nuclei have ground states with spin $J = 0$

The $A^{-1/2}$ dependence reflects that pairing is a surface effect, becoming less important for large nuclei.

5.2.4 Application: Binding Energy Calculations

Example

Calculate the binding energy of ^{56}Fe using the SEMF.

Given: For ^{56}Fe, $A = 56$, $Z = 26$, $N = 30$
Solution:
Step 1: Volume term

$$B_V = a_V A = 15.75 \times 56 = 882.0 \text{ MeV} \tag{5.19}$$

Step 2: Surface term

$$B_S = -a_S A^{2/3} = -17.8 \times (56)^{2/3} = -17.8 \times 14.73 = -262.2 \text{ MeV} \tag{5.20}$$

Step 3: Coulomb term

$$B_C = -a_C \frac{Z^2}{A^{1/3}} = -0.711 \times \frac{(26)^2}{(56)^{1/3}} = -0.711 \times \frac{676}{3.826} = -125.7 \text{ MeV} \tag{5.21}$$

Step 4: Asymmetry term

$$B_A = -a_A \frac{(A-2Z)^2}{A} = -23.7 \times \frac{(56-52)^2}{56} = -23.7 \times \frac{16}{56} = -6.8 \text{ MeV} \tag{5.22}$$

Step 5: Pairing term
Since both $Z = 26$ and $N = 30$ are even (even-even nucleus):

$$\delta = +a_P A^{-1/2} = +11.18 \times (56)^{-1/2} = +11.18 \times 0.134 = +1.5 \text{ MeV} \tag{5.23}$$

Step 6: Total binding energy

$$\begin{aligned} B_{\text{total}} &= B_V + B_S + B_C + B_A + \delta && (5.24)\\ &= 882.0 - 262.2 - 125.7 - 6.8 + 1.5 && (5.25)\\ &= 488.8 \text{ MeV} && (5.26) \end{aligned}$$

Comparison with experiment: The experimental value is $B_{\text{exp}} = 492.3$ MeV.
Error:

$$\text{Error} = \frac{|488.8 - 492.3|}{492.3} \times 100\% = 0.71\% \tag{5.27}$$

This excellent agreement demonstrates the power of the SEMF!

5.2.5 The Binding Energy Curve

One of the most important predictions of the liquid drop model is the shape of the binding energy per nucleon curve.

The binding energy per nucleon is:

$$\frac{B}{A} = a_V - a_S A^{-1/3} - a_C \frac{Z^2}{A^{4/3}} - a_A \frac{(A-2Z)^2}{A^2} + \frac{\delta}{A} \tag{5.28}$$

Key features:

- For small A: Surface term dominates (large $A^{-1/3}$), reducing B/A
- For intermediate A (around $A = 56$): Maximum $B/A \approx 8.8$ MeV
- For large A: Coulomb term grows, reducing B/A

Consequences for nuclear processes:

- **Fusion** of light nuclei releases energy (climbing toward the peak)
- **Fission** of heavy nuclei releases energy (moving toward the peak from above)
- Iron-56 and nickel-58 are among the most stable nuclei

5.2.6 The Line of Beta Stability

For a given mass number A, we can find the most stable value of Z by minimizing the binding energy (or equivalently, minimizing the mass).

Setting $\frac{\partial B}{\partial Z} = 0$:

Mathematical Derivation

$$\frac{\partial}{\partial Z}\left[-a_C\frac{Z^2}{A^{1/3}} - a_A\frac{(A-2Z)^2}{A}\right] = 0 \tag{5.29}$$

$$-2a_C\frac{Z}{A^{1/3}} + 4a_A\frac{(A-2Z)}{A} = 0 \tag{5.30}$$

Solving for Z:

$$Z_{\text{stable}} = \frac{A}{2 + \frac{a_C A^{2/3}}{2a_A}} \tag{5.31}$$

For large A, this gives:

$$Z_{\text{stable}} \approx \frac{A}{2}\left(1 - \frac{a_C A^{2/3}}{4a_A}\right) \tag{5.32}$$

This formula predicts:

- For small A: $Z \approx A/2$ (equal protons and neutrons)
- For large A: $Z < A/2$ (more neutrons needed)
- The neutron excess increases with A

5.2.7 Nuclear Fission and the Liquid Drop Model

The liquid drop model provides physical insight into nuclear fission. When a heavy nucleus absorbs a neutron, it can deform and potentially split.

The **fissility parameter** is defined as:

$$X = \frac{E_{\text{Coulomb}}}{2E_{\text{surface}}} = \frac{a_C Z^2/A^{1/3}}{2a_S A^{2/3}} = \frac{a_C}{2a_S}\frac{Z^2}{A} \tag{5.33}$$

Critical value for spontaneous fission:

$$X_{\text{crit}} \approx 1 \quad \Rightarrow \quad \frac{Z^2}{A} \approx \frac{2a_S}{a_C} \approx 50 \tag{5.34}$$

For $Z^2/A > 50$, the nucleus becomes unstable against spontaneous fission.

5.2.8 Limitations of the Liquid Drop Model

While remarkably successful, the liquid drop model has significant limitations:

1. **Magic Numbers:** Cannot explain the exceptional stability of nuclei with 2, 8, 20, 28, 50, 82, or 126 protons or neutrons
2. **Nuclear Spin:** Provides no information about nuclear angular momentum
3. **Magnetic Moments:** Cannot predict nuclear magnetic moments
4. **Excited States:** Does not describe the discrete energy level structure
5. **Shell Effects:** Misses the quantum mechanical shell structure
6. **Detailed Spectroscopy:** Cannot explain the rich variety of nuclear excitations

These failures led to the development of the nuclear shell model.

5.3 The Nuclear Shell Model

5.3.1 Historical Background and Motivation

In 1949, Maria Goeppert Mayer (working at Argonne National Laboratory) and J. Hans D. Jensen (in Germany) independently proposed the nuclear shell model. They shared the 1963 Nobel Prize in Physics for this groundbreaking work.

The shell model was motivated by several puzzling experimental observations that the liquid drop model could not explain.

The Magic Number Phenomenon

Certain nuclei with specific numbers of protons or neutrons exhibit exceptional properties:

Key Point

Magic Numbers: 2, 8, 20, 28, 50, 82, 126
Nuclei with these numbers of protons or neutrons show:

- Enhanced binding energy
- Greater natural abundance
- Higher first excitation energy
- Lower neutron capture cross-sections
- More stable configurations

Experimental Evidence:

1. **Binding Energy Anomalies:** The binding energy shows local maxima at magic numbers, deviating from the smooth SEMF predictions.
2. **Neutron Separation Energy:** The energy required to remove a neutron from a nucleus shows sharp drops after magic numbers:

$$S_n = B(Z, N) - B(Z, N-1) \tag{5.35}$$

3. **Abundance in Nature:** Elements with magic numbers are significantly more abundant. For example, tin (Z=50) has 10 stable isotopes, far more than its neighbors.

4. **First Excited State Energy:** Magic nuclei have unusually high energies for their first excited states, indicating large energy gaps.

5. **Neutron Capture:** Magic nuclei show sharp minima in neutron capture cross-sections, indicating closed shells.

5.3.2 The Independent Particle Model

Fundamental Assumption

The shell model makes a bold assumption: despite strong nucleon-nucleon interactions, each nucleon can be treated as moving independently in an average potential created by all other nucleons.

This is the **mean-field approximation** or **independent particle model**.

Why does this work?

1. **Pauli Exclusion Principle:** Since nucleons are fermions, most scattering events are forbidden because final states are already occupied. This suppresses the effective interaction strength.

2. **Short-Range Force:** The nuclear force is very short-ranged ($\sim$ 1 fm), so nucleons primarily feel an averaged potential from many neighbors.

3. **High Density:** The dense nuclear environment means the mean field is relatively smooth.

The Nuclear Potential

Each nucleon experiences an effective potential $V(r)$ that depends only on its distance from the nuclear center. Several potential forms have been used:

1. Infinite Square Well:

$$V(r) = \begin{cases} -V_0 & r < R \\ \infty & r \geq R \end{cases} \tag{5.36}$$

Simple but unrealistic (infinite barrier at surface).

2. Finite Square Well:

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r \geq R \end{cases} \tag{5.37}$$

where $V_0 \approx 50$ MeV and $R = r_0 A^{1/3}$ with $r_0 \approx 1.2$ fm.

3. Harmonic Oscillator:

$$V(r) = -V_0 + \frac{1}{2} m\omega^2 r^2 \tag{5.38}$$

Mathematically convenient; energy levels are equally spaced.

4. Woods-Saxon Potential (most realistic):

$$V(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R}{a}\right)} \tag{5.39}$$

where $a \approx 0.5$ fm is the surface diffuseness parameter. This smoothly transitions from $-V_0$ in the interior to 0 outside.

5.3.3 Quantum Numbers for Nuclear States

Each single-particle state in the nucleus is characterized by quantum numbers:

Definition

Nuclear Quantum Numbers:

- n: Principal quantum number (radial nodes: $n-1$)
- ℓ: Orbital angular momentum quantum number
- m_ℓ: Magnetic quantum number ($-\ell \leq m_\ell \leq \ell$)
- s: Intrinsic spin quantum number ($s = 1/2$ for nucleons)
- m_s: Spin projection ($m_s = \pm 1/2$)
- j: Total angular momentum: $j = \ell \pm 1/2$
- m_j: Total angular momentum projection ($-j \leq m_j \leq j$)

Spectroscopic Notation: States are labeled as $n\ell_j$
where ℓ is denoted by letters:

$\ell =$	0	1	2	3	4	5	6	7	8
Symbol	s	p	d	f	g	h	i	j	k

Examples: $1s_{1/2}$, $1p_{3/2}$, $1d_{5/2}$, $2s_{1/2}$, etc.
Degeneracy: Each state with quantum number j can accommodate:

$$2j + 1 \text{ nucleons} \tag{5.40}$$

This accounts for the different m_j projections.

5.3.4 Energy Levels Without Spin-Orbit Coupling

Initially, physicists attempted to explain magic numbers using simple potentials without spin-orbit coupling. For a three-dimensional isotropic harmonic oscillator, the energy levels are:

$$E_{n\ell} = \hbar\omega \left(2n + \ell - \frac{3}{2}\right) \tag{5.41}$$

where the principal quantum number $n = 1, 2, 3, \ldots$ and orbital angular momentum $\ell = 0, 1, 2, \ldots, n-1$.
Alternatively, using $N = 2(n-1) + \ell$ (total oscillator quantum number):

$$E_N = \hbar\omega \left(N + \frac{3}{2}\right), \quad N = 0, 1, 2, 3, \ldots \tag{5.42}$$

States with the same N have the same energy (degeneracy).
Level occupation:

- $N = 0$: $\ell = 0$ (1s), capacity = 2
- $N = 1$: $\ell = 1$ (1p), capacity = 6
- $N = 2$: $\ell = 0, 2$ (2s, 1d), capacity = 2+10 = 12
- $N = 3$: $\ell = 1, 3$ (2p, 1f), capacity = 6+14 = 20

Cumulative totals: 2, 8, 20, 40, 70, ...
This reproduces magic numbers 2, 8, and 20, but fails for 28, 50, 82, and 126!

5.3.5 The Breakthrough: Spin-Orbit Coupling

The key insight by Mayer and Jensen was that a **strong spin-orbit coupling** is necessary to explain all magic numbers.

Spin-Orbit Interaction

The spin-orbit coupling term in the Hamiltonian is:

$$H_{LS} = -Cf(r)\vec{\ell} \cdot \vec{s} \tag{5.43}$$

where:

- C is the coupling strength
- $f(r)$ is a radial function (typically $\propto dV/dr$)
- $\vec{\ell}$ is the orbital angular momentum operator
- $\vec{s}$ is the spin angular momentum operator

Using $\vec{j} = \vec{\ell} + \vec{s}$, we have:

$$\vec{\ell} \cdot \vec{s} = \frac{1}{2}(j^2 - \ell^2 - s^2) = \frac{\hbar^2}{2}[j(j+1) - \ell(\ell+1) - s(s+1)] \tag{5.44}$$

Since $s = 1/2$:

$$\vec{\ell} \cdot \vec{s} = \frac{\hbar^2}{2} \begin{cases} \ell & \text{for } j = \ell + 1/2 \\ -(\ell+1) & \text{for } j = \ell - 1/2 \end{cases} \tag{5.45}$$

Energy splitting:

$$\Delta E_{LS} = -C\langle f(r) \rangle \times \frac{\hbar^2}{2} \begin{cases} \ell & \text{for } j = \ell + 1/2 \\ -(\ell+1) & \text{for } j = \ell - 1/2 \end{cases} \tag{5.46}$$

Crucial point: In nuclei, $C > 0$, which means:

- $j = \ell + 1/2$ has *lower* energy (more bound)
- $j = \ell - 1/2$ has *higher* energy (less bound)

This is **opposite** to atoms! The splitting increases with ℓ:

$$\Delta E = E_{j=\ell-1/2} - E_{j=\ell+1/2} \propto (2\ell + 1) \tag{5.47}$$

For large ℓ, the splitting can be so large that the $j = \ell + 1/2$ state drops below states from higher oscillator shells!

5.3.6 Complete Shell Model Level Scheme

With spin-orbit coupling, the ordering of levels becomes:

State	Capacity (2j+1)	Cumulative	Magic Number
$1s_{1/2}$	2	2	**2**
$1p_{3/2}$	4	6	
$1p_{1/2}$	2	8	**8**
$1d_{5/2}$	6	14	
$2s_{1/2}$	2	16	
$1d_{3/2}$	4	20	**20**
$1f_{7/2}$	8	28	**28**
$2p_{3/2}$	4	32	
$1f_{5/2}$	6	38	
$2p_{1/2}$	2	40	
$1g_{9/2}$	10	50	**50**
$2d_{5/2}$	6	56	
$1g_{7/2}$	8	64	
$3s_{1/2}$	2	66	
$2d_{3/2}$	4	70	
$1h_{11/2}$	12	82	**82**
$2f_{7/2}$	8	90	
$1h_{9/2}$	10	100	
$3p_{3/2}$	4	104	
$2f_{5/2}$	6	110	
$3p_{1/2}$	2	112	
$1i_{13/2}$	14	126	**126**

Notice how large-j states (like $1f_{7/2}$, $1g_{9/2}$, $1h_{11/2}$, $1i_{13/2}$) create shell closures at the magic numbers!

5.3.7 Prediction of Nuclear Spins

The shell model makes definite predictions for nuclear ground state spins.

Rules:

1. **Even-Even Nuclei:** All nucleons are paired. Total angular momentum:

$$J = 0^+ \tag{5.48}$$

(The + indicates positive parity.)

2. **Odd-A Nuclei:** The unpaired nucleon determines the nuclear spin:

$$J = j_{\text{unpaired}} \tag{5.49}$$

3. **Odd-Odd Nuclei:** More complex; unpaired proton and neutron couple. Generally:

$$|j_p - j_n| \leq J \leq j_p + j_n \tag{5.50}$$

Example

Predict the ground state spin of ^{17}O.

Solution:
^{17}O has $Z = 8$ protons and $N = 9$ neutrons.
Proton configuration:

- $1s_{1/2}$: 2 protons (filled)
- $1p_{3/2}$: 4 protons (filled)
- $1p_{1/2}$: 2 protons (filled)

Total: 8 protons (closed shell at magic number)
Neutron configuration:

- $1s_{1/2}$: 2 neutrons (filled)
- $1p_{3/2}$: 4 neutrons (filled)
- $1p_{1/2}$: 2 neutrons (filled)
- $1d_{5/2}$: 1 neutron (unpaired!)

The unpaired neutron is in the $1d_{5/2}$ state.
Prediction: $J = 5/2$
Parity is determined by $(-1)^\ell$ where $\ell = 2$ for d-state, so parity is $(-1)^2 = +$.
Final answer: $J^\pi = 5/2^+$
Experimental result: $J^\pi = 5/2^+$ (Perfect agreement!)

Example

Predict the ground state spin of ^{41}Ca.

Solution:
^{41}Ca has $Z = 20$ protons and $N = 21$ neutrons.
$Z = 20$ is a magic number (closed shell), so protons contribute $J = 0$.
For neutrons: 20 neutrons fill shells up to magic number 20. The 21st neutron goes into the next available state: $1f_{7/2}$.
Prediction: $J^\pi = 7/2^-$ (since $\ell = 3$ for f-state, parity is $(-1)^3 = -$)
Experimental result: $J^\pi = 7/2^-$

5.3.8 Magnetic Moments: Schmidt Lines

Nuclear magnetic moments provide another stringent test of the shell model.

For a single unpaired nucleon (odd-A nucleus), the magnetic moment is:
For odd proton:

$$\mu = g_j \mu_N j \tag{5.51}$$

where the g-factor is:

$$g_j = g_\ell \frac{j(j+1) + \ell(\ell+1) - s(s+1)}{2j(j+1)} + g_s \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)} \tag{5.52}$$

For protons: $g_\ell = 1$, $g_s \approx 5.586$

For neutrons: $g_\ell = 0$, $g_s \approx -3.826$
where $\mu_N = 5.051 \times 10^{-27}$ J/T is the nuclear magneton.
The **Schmidt lines** are the theoretical predictions for $j = \ell \pm 1/2$.
Experimental magnetic moments generally fall between the Schmidt lines, with deviations explained by configuration mixing and collective effects.

5.3.9 Successes of the Shell Model

The nuclear shell model successfully explains:

1. **Magic Numbers:** All observed magic numbers are reproduced
2. **Nuclear Spins:** Accurate predictions for ground state spins
3. **Magnetic Moments:** Qualitative agreement with Schmidt diagrams
4. **Stability Patterns:** Enhanced stability at closed shells
5. **Excitation Energies:** High first excitation energies for magic nuclei
6. **Isomerism:** Explains long-lived excited states

5.3.10 Limitations of the Shell Model

Despite its successes, the shell model has limitations:

1. **Deformed Nuclei:** Cannot explain large quadrupole moments of mid-shell nuclei
2. **Collective Excitations:** Does not describe rotational and vibrational bands
3. **Enhanced E2 Transitions:** Observed transition rates are much larger than single-particle estimates
4. **Many Valence Nucleons:** Calculations become intractable far from closed shells
5. **Backbending:** Cannot explain moment of inertia changes at high spin

These limitations led to the development of the collective model.

5.4 The Collective Model

5.4.1 Motivation and Historical Development

In the early 1950s, it became clear that neither the liquid drop model nor the shell model alone could explain all nuclear phenomena. Aage Bohr (son of Niels Bohr) and Ben Mottelson developed the **collective model** (also called the **unified model**), which synthesizes aspects of both models. They shared the 1975 Nobel Prize in Physics for this work.

Experimental Observations Requiring a New Model

1. **Large Electric Quadrupole Moments:** Many nuclei have quadrupole moments far exceeding single-particle estimates, indicating non-spherical (deformed) shapes.
2. **Rotational Bands:** Sequences of energy levels with spacing proportional to $J(J+1)$, characteristic of rotating systems.

3. **Enhanced E2 Transitions:** Electric quadrupole ($E2$) transition rates between states are enhanced by factors of 10-100 compared to single-particle predictions.
4. **Low-Lying Excited States:** Nuclei far from closed shells have closely spaced low-energy excited states.

5.4.2 Basic Concepts of Collective Motion

The collective model recognizes two types of nuclear motion:

1. **Single-Particle Motion:** Individual nucleons moving in shells (shell model aspect)
2. **Collective Motion:** Coordinated motion of many nucleons acting together (liquid drop aspect)

These motions are coupled and influence each other.

5.4.3 Types of Collective Excitations

Vibrational Motion

Spherical or nearly spherical nuclei can oscillate about their equilibrium shape. The most important modes are:

1. Monopole Vibrations (Breathing Mode):

$$R(\theta, t) = R_0[1 + \alpha_0(t)] \tag{5.53}$$

The nucleus expands and contracts uniformly.

2. Dipole Vibrations:

$$R(\theta, t) = R_0[1 + \alpha_1(t)\cos\theta] \tag{5.54}$$

The center of mass oscillates (usually not excited in ground state nuclei).

3. Quadrupole Vibrations (Most Important):

$$R(\theta, t) = R_0[1 + \alpha_2(t)Y_2^0(\theta)] \tag{5.55}$$

The nucleus oscillates between oblate and prolate shapes.

Quantum Description:

Vibrational excitations are quantized as **phonons**. For quadrupole vibrations:

$$E_n = n\hbar\omega_2, \quad n = 0, 1, 2, 3, \ldots \tag{5.56}$$

Since quadrupole phonons carry angular momentum $\lambda = 2$:

- $n = 0$: Ground state, $J = 0^+$
- $n = 1$: One phonon, $J = 2^+$, Energy $= \hbar\omega_2$
- $n = 2$: Two phonons, $J = 0^+, 2^+, 4^+$, Energy $= 2\hbar\omega_2$
- $n = 3$: Three phonons, $J = 0^+, 2^+, 3^+, 4^+, 6^+$, Energy $= 3\hbar\omega_2$

Characteristic ratio for vibrational nuclei:

$$\frac{E(4_1^+)}{E(2_1^+)} = \frac{2\hbar\omega}{hbar\omega} = 2.0 \tag{5.57}$$

However, anharmonic effects and phonon interactions can modify this.

Example: ^{110}Cd exhibits vibrational spectrum:

- $E(2_1^+) = 0.658$ MeV
- $E(4_1^+) = 1.475$ MeV
- Ratio $= 1.475/0.658 = 2.24$ (close to 2)

Rotational Motion

Permanently deformed nuclei can rotate as rigid or nearly rigid bodies.

Classical Rotational Energy:

$$E_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{L^2}{2\mathcal{I}} \tag{5.58}$$

where $\mathcal{I}$ is the moment of inertia and L is angular momentum.

Quantum Mechanical Version:

$$E_J = \frac{\hbar^2}{2\mathcal{I}}J(J+1) = AJ(J+1) \tag{5.59}$$

where $A = \hbar^2/(2\mathcal{I})$ is the rotational constant.

For even-even nuclei with axial symmetry:

$$J = 0, 2, 4, 6, 8, \ldots \quad \text{(only even values)} \tag{5.60}$$

Characteristic ratio for rotational nuclei:

$$\frac{E(4^+)}{E(2^+)} = \frac{4 \times 5}{2 \times 3} = \frac{20}{6} \approx 3.33 \tag{5.61}$$

This is a clear signature of rotational motion!

Example: ^{238}U exhibits rotational spectrum:

- $E(2^+) = 0.0448$ MeV
- $E(4^+) = 0.148$ MeV
- $E(6^+) = 0.307$ MeV
- $E(8^+) = 0.518$ MeV

Ratios: $E(4^+)/E(2^+) = 3.30$, $E(6^+)/E(2^+) = 6.85$ (close to 7), $E(8^+)/E(2^+) = 11.6$ (close to 12)

These ratios confirm rotational character!

5.4.4 Nuclear Deformation

Parameterization of Nuclear Shape

The surface of a deformed nucleus can be expanded in spherical harmonics:

$$R(\theta, \phi) = R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_\lambda^\mu(\theta, \phi)\right] \tag{5.62}$$

For axially symmetric shapes:

$$R(\theta) = R_0 \left[1 + \beta_2 Y_2^0(\theta) + \beta_4 Y_4^0(\theta) + \ldots\right] \tag{5.63}$$

The **quadrupole deformation parameter** β_2 (often written as β) is most important:

- $\beta > 0$: Prolate deformation (football shape)
- $\beta < 0$: Oblate deformation (disk shape)
- $\beta = 0$: Spherical shape

Typical values: $|\beta| \approx 0.2$ to 0.3 for well-deformed nuclei.

Relationship to Quadrupole Moment

The intrinsic electric quadrupole moment is:

$$Q_0 = \frac{3}{\sqrt{5\pi}} ZeR_0^2\beta \tag{5.64}$$

For large deformations ($\beta \approx 0.3$), Q_0 can be several barns (1 barn = 10^{-28} m^2).

5.4.5 Deformation Systematics

Nuclear deformation varies systematically across the nuclear chart:

- **Near Magic Numbers:** $\beta \approx 0$ (spherical)
- **Mid-Shell Regions:** β reaches maximum ($\beta \approx 0.3$ for rare earths $150 < A < 190$)
- **Transitional Regions:** β varies rapidly

Well-Deformed Regions:

1. $50 < Z < 82$ and $82 < N < 126$ (rare earth nuclei)
2. $Z > 88$ and $N > 126$ (actinide nuclei)

5.4.6 The Nilsson Model

For deformed nuclei, the spherical shell model must be modified. Lars Nilsson developed a model incorporating:

1. Anisotropic Harmonic Oscillator:

$$V(x, y, z) = \frac{1}{2}m[\omega_\perp^2(x^2 + y^2) + \omega_z^2 z^2] \tag{5.65}$$

where $\omega_\perp \neq \omega_z$ for deformed nuclei.

2. Spin-Orbit Coupling:

$$V_{LS} = -C\vec{\ell} \cdot \vec{s} \tag{5.66}$$

3. ℓ^2 Term:

$$V_{\ell^2} = -D\ell^2 \tag{5.67}$$

The Nilsson Hamiltonian:

$$H = T + V_{\text{oscillator}} + V_{LS} + V_{\ell^2} \tag{5.68}$$

The energy levels are plotted as functions of deformation parameter ϵ (related to β). These **Nilsson diagrams** show how spherical shell model states evolve into deformed states.

5.4.7 Rotational Bands

In deformed nuclei, rotational motion builds upon intrinsic single-particle configurations, creating **rotational bands**:

$$E(J) = E_{\text{intrinsic}} + \frac{\hbar^2}{2\mathcal{I}}[J(J+1) - K^2] \tag{5.69}$$

where K is the projection of J on the symmetry axis.

For the ground state band ($K = 0$):

$$E(J) = E_0 + AJ(J+1) \tag{5.70}$$

Moment of Inertia:
The rigid-body moment of inertia for an ellipsoid:

$$\mathcal{I}_{\text{rigid}} = \frac{2}{5} M R_0^2 \beta^2 \tag{5.71}$$

However, observed moments of inertia are typically 40-60% of rigid-body values, indicating the nucleus doesn't rotate as a completely rigid object (some nucleons decouple).

5.4.8 Coupling of Single-Particle and Collective Modes

The collective model recognizes that single-particle and collective motions coexist and couple. The total wavefunction factorizes approximately:

$$\Psi_{\text{total}} = \Psi_{\text{intrinsic}}(\text{internal coords}) \times \Psi_{\text{collective}}(\text{shape coords}) \tag{5.72}$$

This coupling explains:

- Particle-vibration coupling
- Coriolis effects in rotating nuclei
- Band mixing
- Backbending phenomena at high spin

5.5 Comparison and Synthesis of Nuclear Models

5.5.1 Complementary Nature of Models

The three models are not mutually exclusive but complementary:

Table 5.1: Comparison of Nuclear Models

Aspect	**Liquid Drop**	**Shell Model**	**Collective Model**
Basic Picture	Nucleus as incompressible fluid drop	Independent particles in mean field	Single-particle + collective motion
Key Successes	Binding energies, fission, mass trends	Magic numbers, spins, stability	Deformation, rotations, vibrations
Best For	Global properties, systematic trends	Near closed shells	Mid-shell, deformed nuclei
Energy Scale	MeV (bulk binding)	MeV (shell gaps)	keV (collective excitations)
Quantum Effects	Minimal (classical)	Full (single-particle)	Mixed (collective coordinates)
Parameters	5 (SEMF)	Potential depth, radius, spin-orbit	Many (deformation, pairing, etc.)
Limitations	No shell effects, no quantum states	Poor for deformed nuclei	Complex, many parameters

5.5.2 Unified Picture

Modern nuclear structure theory integrates all three perspectives:

1. **Macroscopic-Microscopic Approach:** Combines liquid drop (macroscopic) with shell corrections (microscopic) for accurate mass predictions.
2. **Interacting Boson Model (IBM):** Treats nucleon pairs as bosons, naturally incorporating both vibrational and rotational limits.
3. **Density Functional Theory:** Energy density functionals that include mean-field (shell model) and correlation (collective) effects.
4. **Ab Initio Methods:** For light nuclei, solving the many-body problem from first principles using realistic interactions.

5.5.3 Selection of Appropriate Model

Table 5.2: Model Selection Guide

Nuclear Region/Property	**Appropriate Model**
Global mass trends	Liquid Drop (SEMF)
Magic nuclei properties	Shell Model
Light nuclei ($A < 40$)	Shell Model, Ab initio
Deformed rare earth nuclei	Collective (Nilsson + Rotational)
Vibrational excitations	Collective (Phonon model)
Fission barriers	Liquid Drop + Shell Corrections
High-spin states	Collective (Cranked shell model)
Transition rates	Collective Model

5.6 Summary and Conclusions

In this chapter, we have explored the three fundamental models of nuclear structure, each providing unique insights into the complex behavior of atomic nuclei.

The Liquid Drop Model treats the nucleus as a charged, incompressible fluid drop. Its crowning achievement is the semi-empirical mass formula (SEMF), which accurately predicts binding energies using just five parameters. The model successfully explains:

- Systematic trends in nuclear masses
- The general shape of the binding energy curve
- Nuclear fission and the fissility parameter
- The line of beta stability

However, it cannot account for quantum effects such as magic numbers and nuclear spins.

The Shell Model introduces quantum mechanics into nuclear structure, treating nucleons as independent particles moving in a mean potential. The crucial inclusion of strong spin-orbit coupling explains all observed magic numbers (2, 8, 20, 28, 50, 82, 126). The model successfully predicts:

- Nuclear ground state spins and parities

- Enhanced stability at closed shells
- Magnetic moments (via Schmidt lines)
- First excitation energies

Yet it struggles with deformed nuclei and collective excitations.

The Collective Model synthesizes aspects of both previous models, recognizing that nuclei exhibit both single-particle quantum behavior and collective classical-like motion. It explains:

- Nuclear deformations and quadrupole moments
- Rotational bands in deformed nuclei
- Vibrational excitations in spherical nuclei
- Enhanced electromagnetic transition rates

The model bridges the gap between microscopic and macroscopic descriptions.

Key Point

Fundamental Insight:
No single model explains all nuclear phenomena. The appropriate model depends on:

- The nuclear region (near/far from closed shells)
- The property of interest (mass, spin, excitations)
- The required accuracy

Modern nuclear physics uses elements from all models, recognizing their complementary nature.

5.6.1 Looking Forward

Contemporary nuclear structure research builds upon these classical models:

- High-performance computing enables ab initio calculations for increasingly heavy nuclei
- Radioactive beam facilities explore exotic nuclei far from stability
- Theoretical advances in density functional theory and beyond-mean-field methods
- Experimental discoveries of new magic numbers in neutron-rich nuclei

Yet the conceptual frameworks established by the liquid drop, shell, and collective models remain essential for understanding the atomic nucleus.

5.7 Practice Problems

1. Using the SEMF, calculate the binding energy of ^{208}Pb ($Z = 82$, $N = 126$). Compare with the experimental value of 1636 MeV. What is the percentage error?
2. Explain why ^{16}O ($Z = 8$, $N = 8$) is exceptionally stable, while ^{17}O and ^{15}O are less stable.
3. For ^{141}Pr ($Z = 59$, $N = 82$), use the shell model to predict the ground state spin and parity.

4. A nucleus exhibits excited states at energies: $E(2^+) = 100$ keV, $E(4^+) = 333$ keV, $E(6^+) = 667$ keV. Calculate the ratios and determine if this is better described as vibrational or rotational.

5. Calculate the most stable value of Z for $A = 100$ using the SEMF. Compare with naturally occurring stable isotopes.

6. Explain why nuclei with $50 < Z < 82$ and $82 < N < 126$ are often strongly deformed.

7. If a nucleus has a quadrupole moment $Q_0 = 5$ barns and $Z = 70$, $A = 170$, calculate the deformation parameter β (use $r_0 = 1.2$ fm).

8. Derive the energy ratio $E(6^+)/E(2^+)$ for a pure rotational spectrum.

End of Chapter 5: Nuclear Models

Chapter 6

Nuclear Reactions

6.1 Introduction to Nuclear Reactions

6.1.1 What is a Nuclear Reaction?

A nuclear reaction occurs when an atomic nucleus interacts with another nucleus or subatomic particle, resulting in the production of one or more new nuclides or particles. Unlike chemical reactions, which involve the rearrangement of electrons in atomic orbitals, nuclear reactions involve changes in the nucleus itself and typically release or absorb millions of times more energy.

The general form of a nuclear reaction can be written as:

$$a + X \rightarrow Y + b \tag{6.1}$$

where:

- a is the projectile (incident particle)
- X is the target nucleus
- Y is the product nucleus (residual nucleus)
- b is the emitted particle or radiation

Definition

Nuclear Reaction: A process in which two nuclear particles (typically a nucleus and a subatomic particle, or two nuclei) interact to produce different nuclear particles and/or photons.

6.1.2 Notation for Nuclear Reactions

Nuclear reactions are commonly written in a compact notation:

$$X(a,b)Y \quad \text{or} \quad {}^{A}_{Z}X(a,b){}^{A'}_{Z'}Y \tag{6.2}$$

This represents the reaction: $a + X \rightarrow Y + b$
Examples:

- ${}^{14}\mathrm{N}(n,p){}^{14}\mathrm{C}$ means: $n + {}^{14}\mathrm{N} \rightarrow {}^{14}\mathrm{C} + p$
- ${}^{27}\mathrm{Al}(\alpha,n){}^{30}\mathrm{P}$ means: $\alpha + {}^{27}\mathrm{Al} \rightarrow {}^{30}\mathrm{P} + n$
- ${}^{63}\mathrm{Cu}(p,\gamma){}^{64}\mathrm{Zn}$ means: $p + {}^{63}\mathrm{Cu} \rightarrow {}^{64}\mathrm{Zn} + \gamma$

6.1.3 Historical Perspective

The first artificial nuclear reaction was achieved by Ernest Rutherford in 1919:

$$^{14}\mathrm{N} + \alpha \rightarrow ^{17}\mathrm{O} + p \tag{6.3}$$

This landmark experiment demonstrated that elements could be transmuted, a goal that alchemists had pursued for centuries. Since then, nuclear reactions have become fundamental tools in:

- Understanding nuclear structure
- Producing radioactive isotopes for medicine and industry
- Generating energy through fission and fusion
- Creating new elements
- Analyzing materials (nuclear analytical techniques)

6.2 Types of Nuclear Reactions

Nuclear reactions can be classified based on several criteria. We will examine the major categories based on the nature of the interaction and the energy transfer.

6.2.1 Elastic Scattering

In elastic scattering, the projectile interacts with the target nucleus and is deflected without changing the internal state of either particle. Both kinetic energy and momentum are conserved.

$$a + X \rightarrow a + X \tag{6.4}$$

Characteristics:

- Total kinetic energy is conserved
- No excitation of internal states
- Quantum states of particles unchanged
- Example: $n+^{12}\mathrm{C} \rightarrow n+^{12}\mathrm{C}$

Applications:

- Neutron moderation in nuclear reactors
- Studying nuclear size and shape
- Determining nuclear potentials

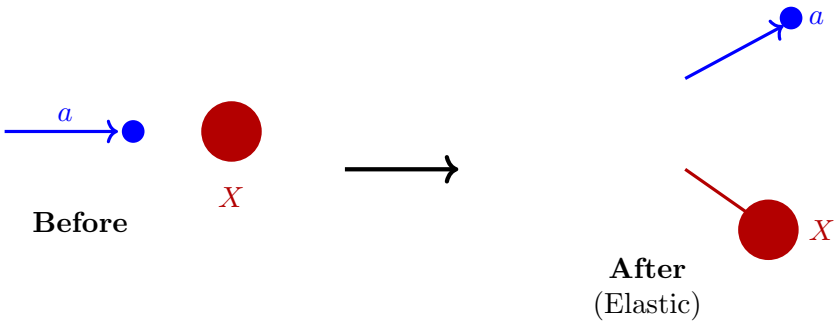


Figure 6.1: Schematic representation of elastic scattering. The projectile a scatters off target X without changing internal states.

6.2.2 Inelastic Scattering

In inelastic scattering, the projectile is scattered and the target nucleus is left in an excited state. The excited nucleus subsequently de-excites by emitting gamma rays.

$$a + X \rightarrow a + X^* \rightarrow a + X + \gamma \tag{6.5}$$

where X^* denotes the excited state.
Characteristics:

- Kinetic energy is not conserved (some converted to excitation)
- Momentum is conserved
- Target nucleus is excited
- Q-value is negative
- Example: n+^{12}C $\rightarrow$ n+^{12}C* $\rightarrow$ n+^{12}C +γ (4.43 MeV)

Applications:

- Studying nuclear energy levels
- Nuclear spectroscopy
- Determining nuclear structure

6.2.3 Capture Reactions

In capture reactions, the projectile is absorbed by the target nucleus, forming a compound nucleus. The compound nucleus then de-excites by emitting radiation.

Radiative Capture

The compound nucleus emits one or more gamma rays:

$$a + X \rightarrow C^* \rightarrow Y + \gamma \tag{6.6}$$

Example:

$$n + {}^{63}\text{Cu} \rightarrow {}^{64}\text{Cu}^* \rightarrow {}^{64}\text{Cu} + \gamma \tag{6.7}$$

This is written in compact form as: $^{63}\text{Cu}(n,\gamma)^{64}\text{Cu}$
Applications:

- Producing radioactive isotopes
- Neutron activation analysis
- Understanding stellar nucleosynthesis

Particle Emission Capture

The compound nucleus emits a different particle:

$$a + X \rightarrow C^* \rightarrow Y + b \tag{6.8}$$

Common types:

- (n, p) reactions: neutron in, proton out
- (n, α) reactions: neutron in, alpha particle out
- (p, n) reactions: proton in, neutron out
- (α, n) reactions: alpha in, neutron out

Example:

$$\alpha + {}^{27}\text{Al} \rightarrow {}^{31}\text{P}^* \rightarrow {}^{30}\text{P} + n \tag{6.9}$$

Written as: $^{27}\text{Al}(\alpha, n)^{30}\text{P}$
This was the reaction used to discover the neutron (Chadwick, 1932):

$$\alpha + {}^{9}\text{Be} \rightarrow {}^{12}\text{C} + n \tag{6.10}$$

6.2.4 Fission Reactions

Nuclear fission is the splitting of a heavy nucleus into two or more lighter nuclei (fission fragments), accompanied by the release of neutrons and a large amount of energy.

$$n + {}^{235}\text{U} \rightarrow \text{Fission Fragments} + \text{neutrons} + \text{energy} \tag{6.11}$$

Example:

$$n + {}^{235}\text{U} \rightarrow {}^{92}\text{Kr} + {}^{141}\text{Ba} + 3n + \sim 200 \text{ MeV} \tag{6.12}$$

Characteristics:

- Typically occurs in heavy nuclei (A ¿ 230)
- Releases enormous energy ($\sim$ 200 MeV per fission)
- Produces 2-3 neutrons on average
- Can be spontaneous or induced
- Asymmetric fission is most common

Fission Energy Distribution:

Component	**Energy (MeV)**
Kinetic energy of fission fragments	167
Kinetic energy of neutrons	5
Prompt gamma rays	7
Beta particles from fission products	7
Neutrinos	10
Delayed gamma rays	6
Total	**202**

Applications:

- Nuclear power reactors
- Nuclear weapons
- Neutron sources
- Production of fission products for research and medicine

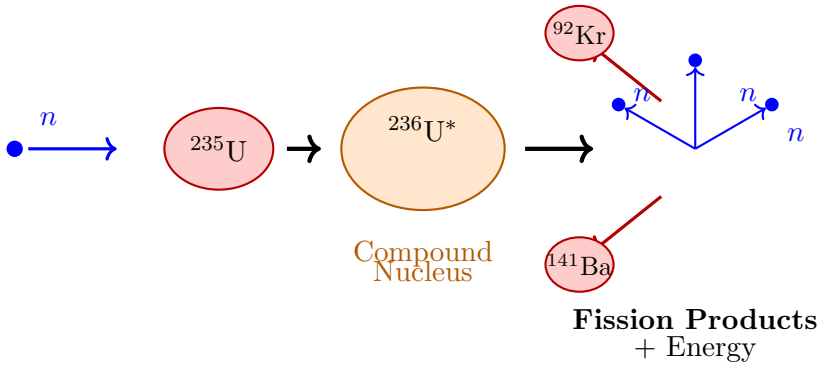


Figure 6.2: Schematic representation of nuclear fission. A neutron induces fission in ^{235}U, producing fission fragments and additional neutrons.

6.2.5 Fusion Reactions

Nuclear fusion is the process where two light nuclei combine to form a heavier nucleus, releasing energy. This is the energy source of stars.

General form:

$$A_1 + A_2 \rightarrow A_3 + \text{particles} + \text{energy} \tag{6.13}$$

Important fusion reactions:

1. Deuterium-Tritium (D-T) reaction:

$$^2\text{H} + ^3\text{H} \rightarrow ^4\text{He} + n + 17.6\ \text{MeV} \tag{6.14}$$

This has the largest cross-section at achievable temperatures.

2. Deuterium-Deuterium (D-D) reactions:

$$^2\text{H} + ^2\text{H} \rightarrow ^3\text{He} + n + 3.27\ \text{MeV} \tag{6.15}$$

$$^2\text{H} + ^2\text{H} \rightarrow ^3\text{H} + p + 4.03\ \text{MeV} \tag{6.16}$$

3. Proton-proton chain (in stars):

$$p + p \rightarrow ^2\text{H} + e^+ + \nu_e \tag{6.17}$$

$$^2\text{H} + p \rightarrow ^3\text{He} + \gamma \tag{6.18}$$

$$^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2p \tag{6.19}$$

Characteristics:

- Requires extremely high temperatures ($\sim 10^7$-10^8 K)
- Coulomb barrier must be overcome
- Releases energy if products have higher binding energy per nucleon

- Powers stars and hydrogen bombs
- Potential future energy source (controlled fusion)

Coulomb Barrier:
For fusion to occur, nuclei must approach to within $\sim 10^{-15}$ m. The Coulomb barrier height is:

$$V_C = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} \tag{6.20}$$

For D-T fusion, $V_C \approx 0.4$ MeV at contact distance.
Applications:

- Stellar energy production
- Thermonuclear weapons
- Experimental fusion reactors (ITER, NIF)
- Neutron production

6.3 Conservation Laws in Nuclear Reactions

Nuclear reactions must obey fundamental conservation laws. These laws allow us to predict possible reactions and calculate reaction energies.

6.3.1 Conservation of Nucleon Number (Baryon Number)

The total number of nucleons (protons + neutrons) must be conserved.

$$A_a + A_X = A_Y + A_b \tag{6.21}$$

where A represents the mass number of each particle.
Example: $^{14}\mathrm{N}(n,p)^{14}\mathrm{C}$

$$1 + 14 = 14 + 1 \quad \checkmark \tag{6.22}$$

6.3.2 Conservation of Charge

The total electric charge must be conserved.

$$Z_a + Z_X = Z_Y + Z_b \tag{6.23}$$

where Z represents the atomic number (number of protons).
Example: $^{14}\mathrm{N}(n,p)^{14}\mathrm{C}$

$$0 + 7 = 6 + 1 \quad \checkmark \tag{6.24}$$

6.3.3 Conservation of Energy

The total energy (mass-energy + kinetic energy) is conserved.

$$(m_a c^2 + T_a) + (m_X c^2 + T_X) = (m_Y c^2 + T_Y) + (m_b c^2 + T_b) \tag{6.25}$$

This can be rearranged to define the **Q-value** of the reaction:

Important Formula

Q-Value Definition:

$$Q = (m_a + m_X - m_Y - m_b)c^2 = T_Y + T_b - T_a - T_X \tag{6.26}$$

The Q-value represents the energy released (if $Q > 0$) or absorbed (if $Q < 0$) in the reaction.

Classification by Q-value:

- **Exothermic (Exoergic):** $Q > 0$ (energy released)
- **Endothermic (Endoergic):** $Q < 0$ (energy absorbed)

For a stationary target ($T_X = 0$):

$$Q = T_Y + T_b - T_a \tag{6.27}$$

6.3.4 Conservation of Momentum

The total linear momentum must be conserved.

$$\boldsymbol{p}_a + \boldsymbol{p}_X = \boldsymbol{p}_Y + \boldsymbol{p}_b \tag{6.28}$$

For a stationary target:

$$\boldsymbol{p}_a = \boldsymbol{p}_Y + \boldsymbol{p}_b \tag{6.29}$$

6.3.5 Conservation of Angular Momentum

The total angular momentum (orbital + spin) must be conserved.

$$\boldsymbol{J}_a + \boldsymbol{J}_X = \boldsymbol{J}_Y + \boldsymbol{J}_b + \boldsymbol{L} \tag{6.30}$$

where $\boldsymbol{L}$ is the orbital angular momentum of the products.
This conservation law restricts possible transitions and affects reaction probabilities.

6.3.6 Conservation of Parity

Parity must be conserved in strong and electromagnetic interactions (but not in weak interactions).

$$\pi_a \cdot \pi_X = \pi_Y \cdot \pi_b \cdot (-1)^L \tag{6.31}$$

where π represents the intrinsic parity and L is the orbital angular momentum quantum number.

6.4 Reaction Kinematics and Q-Value

6.4.1 The Q-Value Equation

The Q-value is the most important parameter characterizing a nuclear reaction. It determines whether a reaction is energetically possible and how much energy is released or required.

Mathematical Derivation

Starting from energy conservation:

$$m_a c^2 + T_a + m_X c^2 + T_X = m_Y c^2 + T_Y + m_b c^2 + T_b \tag{6.32}$$

Rearranging:

$$Q = (m_a + m_X - m_Y - m_b)c^2 = (T_Y + T_b) - (T_a + T_X) \tag{6.33}$$

The Q-value can be calculated from:

1. **Mass differences:**
$$Q = \Delta m \cdot c^2 \tag{6.34}$$
2. **Atomic masses:**
$$Q = [M(a) + M(X) - M(Y) - M(b)] \times 931.5 \text{ MeV} \tag{6.35}$$
where masses are in atomic mass units (u).
3. **Binding energies:**
$$Q = [B(Y) + B(b) - B(a) - B(X)] \tag{6.36}$$

Example

Calculate the Q-value for the reaction: $^{14}\mathbf{N}(n,p)^{14}\mathbf{C}$

Given atomic masses:

- $M(\text{n}) = 1.008665$ u
- $M(^{14}\text{N}) = 14.003074$ u
- $M(^{14}\text{C}) = 14.003242$ u
- $M(\text{p}) = 1.007825$ u

Solution:

$$\begin{aligned} Q &= [M(n) + M(^{14}\text{N}) - M(^{14}\text{C}) - M(p)] \times 931.5 \text{ MeV/u} && (6.37)\\ &= [1.008665 + 14.003074 - 14.003242 - 1.007825] \times 931.5 && (6.38)\\ &= [0.000672] \times 931.5 && (6.39)\\ &= 0.626 \text{ MeV} && (6.40) \end{aligned}$$

Since $Q > 0$, this is an **exothermic reaction** that releases 0.626 MeV.

6.4.2 Kinematics of Two-Body Reactions

For a reaction $a + X \to Y + b$, we need to determine the energies and momenta of the products.

Laboratory Frame (Stationary Target)

In the laboratory frame, the target X is at rest.

Conservation of momentum:

$$\boldsymbol{p}_a = \boldsymbol{p}_Y + \boldsymbol{p}_b \tag{6.41}$$

Conservation of energy:

$$T_a + Q = T_Y + T_b \tag{6.42}$$

For non-relativistic particles:

$$T = \frac{p^2}{2m} \tag{6.43}$$

Energy of products:

Using momentum and energy conservation, the kinetic energy of particle b emitted at angle θ is:

$$T_b = \frac{(m_a m_b)/(m_Y + m_b)}{m_a + m_X} T_a + \frac{m_Y}{m_Y + m_b} Q + 2\sqrt{\frac{m_a m_b m_Y}{(m_Y + m_b)^2 (m_a + m_X)} T_a Q}\cos\theta \tag{6.44}$$

For $Q = 0$ (elastic scattering), this simplifies considerably.

Center of Mass Frame

In the center of mass (CM) frame, the total momentum is zero.

Advantages of CM frame:

- Simpler mathematics
- Physical insight more transparent
- Particles move in opposite directions

Transformation to CM frame:

The CM velocity is:

$$v_{CM} = \frac{m_a v_a + m_X v_X}{m_a + m_X} = \frac{m_a v_a}{m_a + m_X} \tag{6.45}$$

Available energy in CM frame:

The kinetic energy available for the reaction (excluding CM motion):

$$T_{CM} = \frac{m_X}{m_a + m_X} T_a \tag{6.46}$$

This is also called the **available energy**.

6.5 Threshold Energy and Excitation Functions

6.5.1 Threshold Energy

For endothermic reactions ($Q < 0$), there is a minimum kinetic energy of the projectile required for the reaction to occur. This is called the **threshold energy**.

Definition

Threshold Energy (T_{th}): The minimum kinetic energy of the projectile (in the laboratory frame with stationary target) required for an endothermic nuclear reaction to occur.

Mathematical Derivation

At threshold, the products are created at rest in the CM frame (minimum energy configuration). All energy goes into creating mass.
Using energy-momentum conservation in the CM frame:

$$E_{\text{initial}}^2 - p_{\text{initial}}^2 c^2 = E_{\text{final}}^2 \tag{6.47}$$

For a stationary target:

$$(T_a + m_a c^2 + m_X c^2)^2 - p_a^2 c^2 = (m_Y + m_b)^2 c^4 \tag{6.48}$$

Using $p_a^2 c^2 = T_a^2 + 2T_a m_a c^2$ (non-relativistic approximation):
After algebraic manipulation:

$$T_{th} = -Q\left(1 + \frac{m_Y + m_b}{2m_X} + \frac{m_Y + m_b - m_a}{2m_X}\right) \tag{6.49}$$

For $m_a \approx m_b$ and $m_X \approx m_Y$ (common case):

$$T_{th} \approx -Q\left(1 + \frac{|Q|}{2m_X c^2}\right) \tag{6.50}$$

Simplified form (often used):

$$T_{th} = -Q\frac{m_a + m_X + m_Y + m_b}{2m_X} \tag{6.51}$$

Important Formula

Threshold Energy (General Formula):

$$T_{th} = \frac{[(m_Y + m_b)^2 - (m_a + m_X)^2]c^2}{2m_X} \tag{6.52}$$

For $Q < 0$:

$$T_{th} = -Q\left[\frac{(m_Y + m_b)^2 - (m_a + m_X)^2}{2m_X(m_Y + m_b - m_a - m_X)}\right] \tag{6.53}$$

Approximate form:

$$T_{th} \approx -Q\left(1 + \frac{m_a + m_b + m_Y}{2m_X}\right) \quad \text{for } |Q| \ll mc^2 \tag{6.54}$$

Example

Calculate the threshold energy for: $^{16}\mathbf{O}(p,n)^{16}\mathbf{F}$

Given:

- $Q = -10.24$ MeV (endothermic)
- $m_p \approx 1$ u, $m_n \approx 1$ u, $m_X = 16$ u, $m_Y = 16$ u

Solution:
Using the simplified formula:

$$T_{th} = -Q\left(1 + \frac{m_p + m_n + m_Y}{2m_X}\right) \tag{6.55}$$

$$= 10.24\left(1 + \frac{1+1+16}{2\times 16}\right) \tag{6.56}$$

$$= 10.24\left(1 + \frac{18}{32}\right) \tag{6.57}$$

$$= 10.24 \times 1.5625 \tag{6.58}$$

$$= 16.0 \text{ MeV} \tag{6.59}$$

A proton must have at least 16.0 MeV of kinetic energy for this reaction to occur.

6.5.2 Excitation Functions

An **excitation function** is a plot of the reaction cross-section as a function of the projectile energy.

Definition

Excitation Function: A graph showing how the probability of a nuclear reaction (measured by its cross-section) varies with the energy of the incident particle.

Key features:

- For endothermic reactions: $\sigma = 0$ for $T < T_{th}$
- Near threshold: cross-section rises rapidly
- At higher energies: may show resonances (peaks)
- Very high energies: cross-section typically decreases

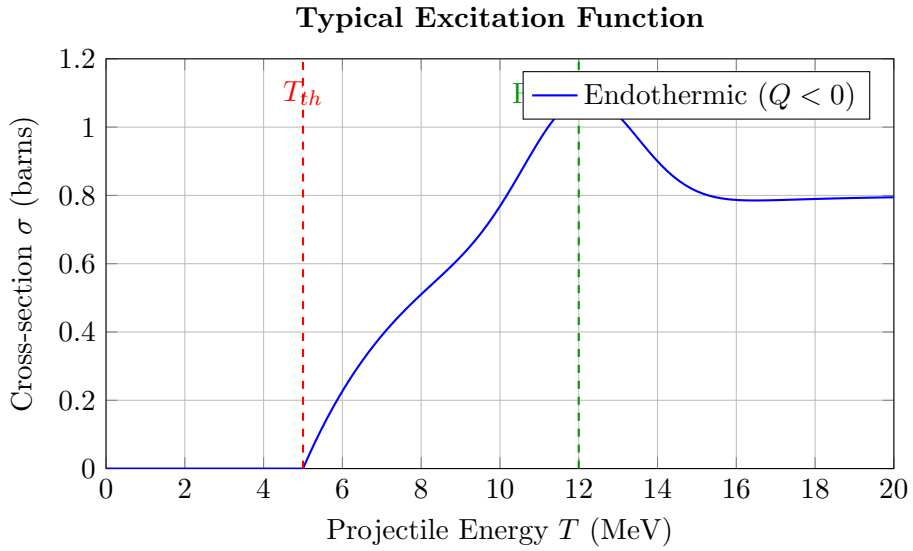


Figure 6.3: Typical excitation function for an endothermic nuclear reaction showing threshold energy and resonance behavior.

6.6 Cross-Section and Reaction Rate

6.6.1 Nuclear Cross-Section

The **cross-section** is a measure of the probability that a specific nuclear reaction will occur. It has dimensions of area.

Definition

Nuclear Cross-Section (σ): A measure of the effective area presented by a target nucleus to an incident particle for a specific reaction. It quantifies the probability of interaction.

Units:

- 1 barn $= 10^{-24}$ cm^2 $= 10^{-28}$ m^2
- Typical nuclear cross-sections: 10^{-3} to 10^{3} barns

6.6.2 Types of Cross-Sections

1. Total Cross-Section (σ_T):

$$\sigma_T = \sigma_{\text{elastic}} + \sigma_{\text{reaction}} \tag{6.60}$$

2. Reaction Cross-Section (σ_R):
Sum of all inelastic processes:

$$\sigma_R = \sigma_{(n,p)} + \sigma_{(n,\gamma)} + \sigma_{(n,2n)} + \dots \tag{6.61}$$

3. Partial Cross-Section:
Cross-section for a specific reaction channel.
4. Differential Cross-Section ($d\sigma/d\Omega$):
Cross-section per unit solid angle:

$$\frac{d\sigma}{d\Omega}(\theta, \phi) \tag{6.62}$$

The total cross-section is obtained by integrating:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \tag{6.63}$$

6.6.3 Relationship to Reaction Probability

Consider a beam of particles incident on a thin target:

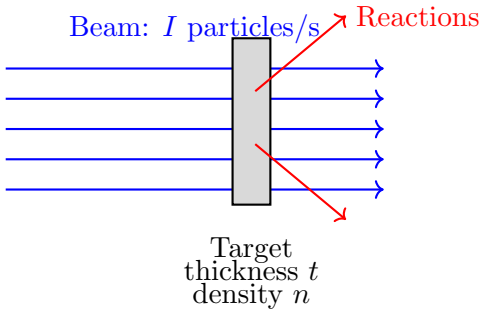


Figure 6.4: Beam of particles incident on a target of thickness t and number density n.

The number of reactions per second is:

$$R = I \cdot n \cdot t \cdot \sigma \cdot A \tag{6.64}$$

where:

- I = incident beam intensity (particles per unit area per unit time)
- n = number density of target nuclei (nuclei per unit volume)
- t = target thickness
- σ = cross-section
- A = beam area

Fraction of particles that react:

$$f = \frac{R}{I \cdot A} = n\sigma t \tag{6.65}$$

For thin targets ($n\sigma t \ll 1$), this fraction is small.

6.6.4 Reaction Rate

The **reaction rate** is the number of reactions occurring per unit time.

Important Formula

Reaction Rate Formula:
For a beam on a target:

$$R = \Phi \cdot N \cdot \sigma \tag{6.66}$$

where:

- R = reaction rate (reactions per second)
- Φ = particle flux (particles per cm^2 per second)
- N = number of target nuclei per cm^2 (areal density)
- σ = cross-section (cm^2)

For a volume with uniform particle densities:

$$R = n_a \cdot n_X \cdot v\sigma \cdot V \tag{6.67}$$

where:

- n_a = projectile number density
- n_X = target number density
- $v\sigma$ = velocity-averaged cross-section (reactivity)
- V = volume

6.6.5 Energy Dependence of Cross-Sections

Cross-sections vary dramatically with energy:

1. Low Energy (Thermal Neutrons):
For many reactions, $\sigma \propto 1/v$ (or $\sigma \propto 1/\sqrt{E}$):

$$\sigma(E) = \sigma_0 \sqrt{\frac{E_0}{E}} \tag{6.68}$$

This is the **1/v law**.

2. Resonance Region:
At specific energies, cross-sections show sharp peaks (resonances) described by the **Breit-Wigner formula**:

$$\sigma(E) = \sigma_0 \frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4} \tag{6.69}$$

where E_R is the resonance energy and Γ is the width.

3. High Energy:
Cross-sections typically decrease and approach geometric values:

$$\sigma_{\text{geo}} \approx \pi R^2 \approx \pi r_0^2 (A_a^{1/3} + A_X^{1/3})^2 \tag{6.70}$$

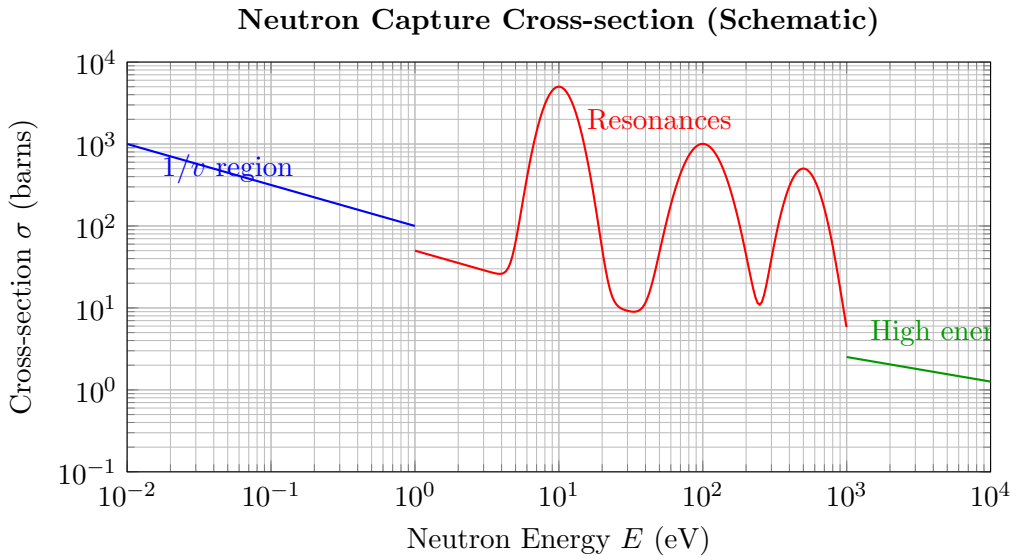


Figure 6.5: Schematic neutron capture cross-section showing 1/v behavior at low energy, resonances at intermediate energies, and decreasing cross-section at high energy.

6.7 Reaction Mechanisms: Compound Nucleus and Direct Reactions

Nuclear reactions proceed through different mechanisms depending on the energy and nature of the interacting particles. The two main categories are compound nucleus reactions and direct reactions.

6.7.1 Compound Nucleus Mechanism

Proposed by Niels Bohr in 1936, the compound nucleus model describes a two-stage reaction process.

Definition

Compound Nucleus: An intermediate nuclear system formed when a projectile is absorbed by a target nucleus. The compound nucleus exists for a relatively long time ($\sim 10^{-16}$ to 10^{-14} s) before decaying.

The Two-Stage Process

Stage 1: Formation

$$a + X \rightarrow C^* \tag{6.71}$$

The projectile is captured, forming an excited compound nucleus C^*.

Stage 2: Decay

$$C^* \rightarrow Y + b \tag{6.72}$$

After a "long" time (many nuclear periods), the compound nucleus decays.

Complete reaction:

$$a + X \rightarrow [C^*] \rightarrow Y + b \tag{6.73}$$

Key Point

Bohr Independence Hypothesis:
The decay of the compound nucleus is **independent** of how it was formed. The compound nucleus "forgets" its formation mode and decays according to statistical laws based only on its energy, angular momentum, and parity.
Mathematically:

$$\sigma_{a,b} = \sigma_{\text{formation}}(a) \times P_{\text{decay}}(b) \tag{6.74}$$

The cross-section factors into formation and decay probabilities.

Characteristics of Compound Nucleus Reactions

1. **Long Lifetime:** $\tau \sim 10^{-16}$ to 10^{-14} s

   (Compare to direct interaction time: $\sim 10^{-22}$ s)

2. **Statistical Decay:** Energy is shared among many nucleons; decay is statistical

3. **Isotropic Emission:** Products are emitted isotropically in the CM frame (no memory of entrance channel)

4. **Resonances:** Cross-sections show resonances corresponding to compound nucleus energy levels

5. **Energy Sharing:** Excitation energy is distributed among all nucleons

Energy Level Density

The number of available states in the compound nucleus increases rapidly with excitation energy:

$$\rho(E) \propto \exp\left(2\sqrt{aE}\right) \tag{6.75}$$

where $a \approx A/8$ MeV^{-1} is the level density parameter.
For $A = 100$ and $E = 10$ MeV:

$$\rho(10 \text{ MeV}) \sim 10^5 \text{ levels per MeV} \tag{6.76}$$

This high density ensures statistical equilibration.

Figure 6.6: The two-stage compound nucleus reaction mechanism: (1) Formation of excited compound nucleus C^*, (2) Statistical decay to products.

6.7.2 Direct Reactions

Direct reactions occur when the projectile interacts with only a few nucleons near the nuclear surface, without forming a long-lived compound nucleus.

Definition

Direct Reaction: A nuclear reaction in which the projectile interacts with the target nucleus through a rapid, localized interaction without complete energy equilibration. Interaction time: $\sim 10^{-22}$ s.

Types of Direct Reactions

1. Elastic and Inelastic Scattering

$$a + X \to a + X \quad \text{or} \quad a + X^* \tag{6.77}$$

2. Stripping Reactions
The projectile "deposits" one or more nucleons in the target:

$$(d, p): \quad d + X \to p + (X + n) \tag{6.78}$$

Example: $^{16}\mathrm{O}(d, p)^{17}\mathrm{O}$
The neutron from deuteron is "stripped off" and captured.
3. Pickup Reactions
The projectile "picks up" one or more nucleons from the target:

$$(p, d): \quad p + X \to d + (X - n) \tag{6.79}$$

Example: $^{17}\mathrm{O}(p, d)^{16}\mathrm{O}$
4. Knockout Reactions
A nucleon is directly knocked out:

$$(p, 2p): \quad p + X \to 2p + (X - p) \tag{6.80}$$

5. Charge Exchange

$$(p, n): \quad p + X \to n + Y \tag{6.81}$$

where $Z_Y = Z_X + 1$, $A_Y = A_X$.

Characteristics of Direct Reactions

1. **Short Interaction Time:** $\tau \sim 10^{-22}$ s (nuclear crossing time)
2. **Peripheral Interaction:** Occurs at nuclear surface
3. **Forward Peaking:** Angular distributions peak in the forward direction
4. **Selective:** Populate specific states (not statistical)
5. **High Energy:** More probable at higher energies
6. **Angular Momentum Matching:** Strong dependence on orbital angular momentum transfer

6.7.3 Comparison of Reaction Mechanisms

Table 6.1: Comparison of Compound Nucleus and Direct Reactions

Property	**Compound Nucleus**	**Direct Reaction**
Interaction Time	Long (10^{-16}-10^{-14} s)	Short ($\sim 10^{-22}$ s)
Energy Sharing	Complete equilibration among all nucleons	Few nucleons involved
Angular Distribution	Isotropic (symmetric) in CM frame	Anisotropic (forward peaked)
Decay Mode	Statistical, memory-less	Direct, remembers entrance channel
Excitation Energy	Broad energy range of products	Specific final states
Cross-section vs E	Shows resonances	Smooth variation
Typical Projectile Energy	Low to medium (¡ 20 MeV)	Medium to high (¿ 10 MeV)
Region of Interaction	Entire nuclear volume	Nuclear surface
Theoretical Description	Statistical model, Hauser-Feshbach	DWBA, optical model
Example	$^{63}\mathrm{Cu}(n,\gamma)^{64}\mathrm{Cu}$	$^{16}\mathrm{O}(d,p)^{17}\mathrm{O}$

6.7.4 Competition Between Mechanisms

In reality, both mechanisms can contribute to a reaction. The relative importance depends on:

- **Projectile Energy:** Higher energy favors direct reactions
- **Projectile Type:** Complex particles (d, α) favor direct reactions
- **Target Mass:** Heavy targets favor compound nucleus formation
- **Q-value:** Large positive Q-values favor compound nucleus

Total cross-section:

$$\sigma_{\text{total}} = \sigma_{\text{compound}} + \sigma_{\text{direct}} \tag{6.82}$$

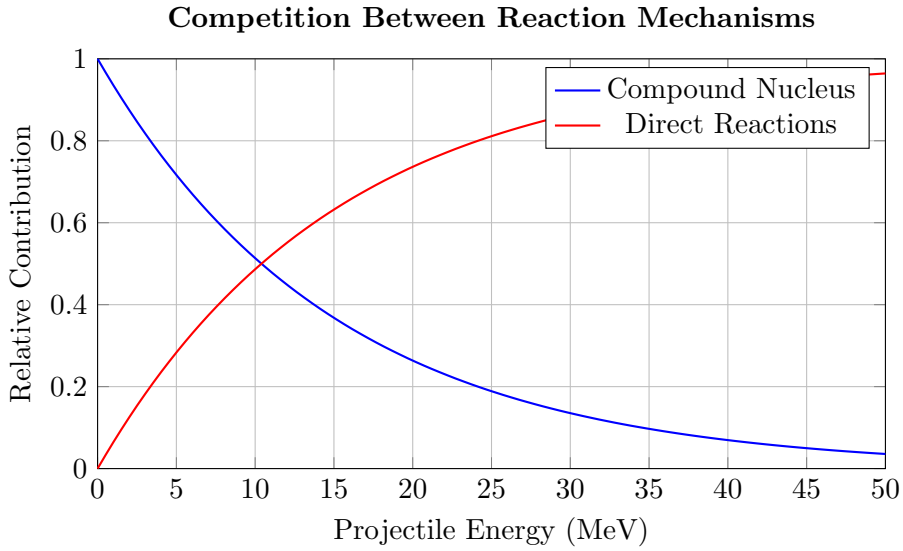


Figure 6.7: Schematic showing the relative contributions of compound nucleus and direct reaction mechanisms as a function of projectile energy.

6.8 Applications of Nuclear Reactions

Nuclear reactions have numerous practical applications:

6.8.1 Energy Production

1. Nuclear Fission Reactors:

- Commercial power generation
- Currently provides ∼10% of world electricity
- Based on ^{235}U and ^{239}Pu fission

2. Nuclear Fusion (Future):

- ITER project (International Thermonuclear Experimental Reactor)
- National Ignition Facility (NIF)
- Potential clean, abundant energy source

6.8.2 Medical Applications

1. Radioisotope Production:

- ^{99m}Tc for medical imaging (via $^{98}\text{Mo}(n,\gamma)^{99}\text{Mo} \rightarrow {}^{99m}\text{Tc}$)
- ^{18}F for PET scans (via $^{18}\text{O}(p,n)^{18}\text{F}$)

2. Cancer Therapy:

- Proton therapy (direct reactions)
- Neutron capture therapy (BNCT)

6.8.3 Materials Analysis

1. Neutron Activation Analysis (NAA):

- Non-destructive elemental analysis
- Detection limits: ppb to ppt
- Used in archaeology, forensics, environmental monitoring

2. Ion Beam Analysis:

- Rutherford Backscattering Spectrometry (RBS)
- Nuclear Reaction Analysis (NRA)
- Particle Induced X-ray Emission (PIXE)

6.8.4 Nuclear Structure Research

- Determining nuclear energy levels
- Measuring nuclear spins and parities
- Studying exotic nuclei far from stability
- Understanding nuclear forces

6.8.5 Astrophysics

- Understanding stellar energy production
- Nucleosynthesis of elements
- Supernova mechanisms
- Big Bang nucleosynthesis

6.9 Summary

In this chapter, we have explored the fundamental aspects of nuclear reactions:

1. **Types of Reactions:** We classified nuclear reactions into elastic scattering, inelastic scattering, capture, fission, and fusion, each with distinct characteristics and applications.
2. **Conservation Laws:** Nuclear reactions must conserve nucleon number, charge, energy, momentum, angular momentum, and parity. These laws constrain possible reactions and allow calculation of reaction energetics.
3. **Q-Value and Kinematics:** The Q-value determines whether a reaction is exothermic or endothermic. Reaction kinematics in laboratory and center-of-mass frames allow calculation of product energies and angles.
4. **Threshold Energy:** Endothermic reactions require a minimum projectile energy (threshold energy) to proceed. We derived the threshold energy formula and applied it to examples.

5. **Cross-Section and Reaction Rate:** The cross-section quantifies the probability of a reaction occurring. Reaction rates depend on particle flux, target density, and cross-section. Cross-sections vary with energy, showing 1/v behavior, resonances, and geometric limiting values.

6. **Reaction Mechanisms:** Nuclear reactions proceed via compound nucleus formation (statistical, isotropic decay) or direct reactions (fast, selective, forward-peaked). The dominant mechanism depends on energy, projectile type, and target properties.

Key Point

Key Takeaways:

- Nuclear reactions transform nuclei and release/absorb large amounts of energy
- Conservation laws govern all nuclear reactions
- Q-value determines reaction energetics: $Q > 0$ (exothermic), $Q < 0$ (endothermic)
- Threshold energy exists for endothermic reactions
- Cross-section measures reaction probability; varies strongly with energy
- Compound nucleus: statistical, slow; Direct reactions: selective, fast
- Applications span energy production, medicine, materials science, and fundamental research

Understanding nuclear reactions is essential for applications ranging from nuclear power and medical isotope production to astrophysical processes and fundamental nuclear physics research. The concepts developed in this chapter form the foundation for advanced studies in nuclear science and technology.

6.10 Practice Problems

1. Calculate the Q-value for the reaction $^{7}\mathrm{Li}(p,n)^{7}\mathrm{Be}$ given:

   - $M(\mathrm{p}) = 1.007825$ u
   - $M(^{7}\mathrm{Li}) = 7.016003$ u
   - $M(^{7}\mathrm{Be}) = 7.016930$ u
   - $M(\mathrm{n}) = 1.008665$ u

   Is this reaction exothermic or endothermic?

2. For the reaction $^{12}\mathrm{C}(p,n)^{12}\mathrm{N}$ with $Q = -18.1$ MeV, calculate the threshold energy for the proton.

3. A beam of 10^{10} protons per second hits a $^{63}\mathrm{Cu}$ target of thickness 1 mm. The density of copper is 8.96 g/cm^3 and the (p,n) reaction cross-section at this energy is 0.5 barns. Calculate the reaction rate.

4. Show that for elastic scattering of a particle of mass m from a stationary target of mass M, the maximum fractional energy loss of the projectile is:

$$\frac{\Delta T_{\max}}{T} = \frac{4mM}{(m+M)^2}$$

5. The compound nucleus $^{64}\text{Zn}^*$ can be formed by:

   (a) n+^{63}Zn

   (b) p+^{63}Cu

   (c) α+^{60}Ni

   According to the compound nucleus model, what determines the relative probability of decay to different exit channels?

6. Explain why direct reactions are more probable at high energies while compound nucleus reactions dominate at low energies.

7. A neutron with kinetic energy 1 MeV undergoes the reaction n+^{14}N $\rightarrow$ p+^{14}C with $Q = +0.626$ MeV. Calculate the maximum kinetic energy the proton can have (consider conservation of energy and momentum).

8. Explain the physical origin of resonances in neutron capture cross-sections and why they become broader and more closely spaced at higher energies.

End of Chapter 6: Nuclear Reactions

Chapter 7

Nuclear Fission and Fusion

7.1 Introduction

Nuclear fission and fusion are two fundamental processes that release enormous amounts of energy through the transformation of atomic nuclei. These processes represent opposite ends of the nuclear energy spectrum: fission involves the splitting of heavy nuclei into lighter fragments, while fusion combines light nuclei to form heavier ones. Both processes release energy because they move nuclei toward the region of maximum binding energy per nucleon, near iron-56 on the nuclear stability curve.

The discovery and harnessing of these processes have profoundly impacted human civilization, from providing abundant electrical power to creating devastating weapons, and understanding stellar energy production. In this chapter, we will explore the physics underlying these phenomena, their applications, and the technological challenges involved in their controlled use.

Historical Note

Discovery of Nuclear Fission
Nuclear fission was discovered in December 1938 by German chemists Otto Hahn and Fritz Strassmann, who found that uranium bombarded with neutrons produced barium, an element with roughly half the atomic mass of uranium. The physical explanation was provided by Lise Meitner and Otto Frisch in early 1939, who recognized that the uranium nucleus had split into two roughly equal parts, releasing enormous energy in the process. This discovery came just months before the outbreak of World War II and led directly to the development of nuclear weapons and nuclear power.

7.2 Nuclear Fission

7.2.1 The Fission Process

Nuclear fission is the process in which a heavy nucleus splits into two or more lighter nuclei (called fission fragments), along with the emission of neutrons and the release of a large amount of energy.

Definition

Nuclear Fission: A nuclear reaction in which a heavy atomic nucleus splits into two or more lighter nuclei (fission fragments), accompanied by the release of neutrons, gamma rays, and a large amount of kinetic energy.

The most important fissionable isotopes for practical applications are:

- **^{235}U:** Naturally occurring (0.72% abundance), fissile
- **^{238}U:** Naturally occurring (99.27% abundance), fissionable but not fissile
- **^{239}Pu:** Artificial, produced in reactors, fissile
- **^{233}U:** Artificial, produced from thorium, fissile

Fissile vs. Fissionable:

- **Fissile:** Can undergo fission with thermal (slow) neutrons
- **Fissionable:** Can undergo fission, but requires fast neutrons

7.2.2 Mechanism of Fission

The fission process can be understood through the liquid drop model with quantum mechanical corrections.

The Liquid Drop Model and Fission

When a neutron is absorbed by a heavy nucleus like ^{235}U, a compound nucleus is formed:

$$n + {}^{235}\text{U} \rightarrow {}^{236}\text{U}^* \rightarrow \text{Fission Products} + \text{neutrons} + \text{energy} \tag{7.1}$$

The excitation energy comes from the neutron binding energy (about 6.5 MeV for ^{235}U).
The fission barrier can be understood by examining the competition between:

- **Surface Energy:** Opposes deformation (increases with deformation)
- **Coulomb Energy:** Favors fission (decreases with deformation)

Fissility Parameter:
The dimensionless fissility parameter Z (also written as X) determines how easily a nucleus undergoes fission:

$$Z = \frac{Z^2/A}{50.88} \tag{7.2}$$

or more accurately:

$$X = \frac{E_C}{2E_S} = \frac{Z^2}{A} \times \frac{a_C}{2a_S} \tag{7.3}$$

where E_C is the Coulomb energy and E_S is the surface energy.
Critical values:

- $Z < 1$: Nucleus is stable against spontaneous fission
- $Z \approx 1$: Fission barrier nearly vanishes
- For ^{235}U: $Z = 235^2/92 \approx 35.6$ (requires activation energy)
- For very heavy elements ($Z > 100$): spontaneous fission becomes dominant

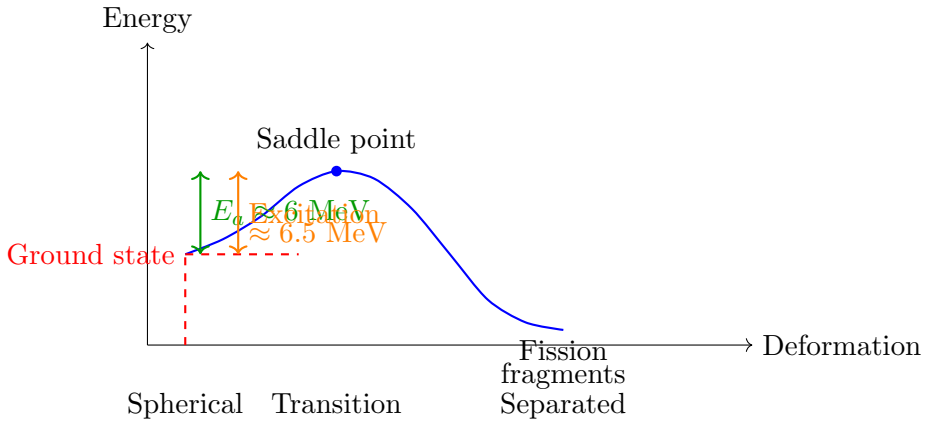


Figure 7.1: Potential energy as a function of nuclear deformation for ^{235}U. The excitation energy from thermal neutron capture (∼6.5 MeV) exceeds the activation energy (∼6 MeV), allowing fission to proceed.

7.2.3 Energetics of Fission

Energy Release in Fission

A typical fission reaction of ^{235}U releases approximately 200 MeV of energy:

$$n + {}^{235}\text{U} \rightarrow {}^{236}\text{U}^* \rightarrow \text{FF}_1 + \text{FF}_2 + 2\text{-}3\,n + \gamma + \sim 200 \text{ MeV} \tag{7.4}$$

Example reaction:

$$n + {}^{235}\text{U} \rightarrow {}^{141}\text{Ba} + {}^{92}\text{Kr} + 3n + 200 \text{ MeV} \tag{7.5}$$

Energy Distribution:

Table 7.1: Energy Distribution in ^{235}U Fission

Component	Energy (MeV)	Percentage (%)
Kinetic energy of fission fragments	167	83.5
Kinetic energy of prompt neutrons	5	2.5
Energy of prompt γ-rays	7	3.5
Energy of fission product decay (β)	7	3.5
Energy of fission product decay (γ)	6	3.0
Neutrino energy (not recoverable)	10	5.0
Total	**202**	**100**

Recoverable energy: ∼192 MeV per fission (neutrinos escape)

Mathematical Derivation

Calculation of Energy Release from Binding Energy:
The energy released can be estimated from the binding energy curve. The binding energy per nucleon for:

- ^{235}U: $B/A \approx 7.6$ MeV
- Fission fragments (A ≈ 95-140): $B/A \approx 8.4$ MeV

Energy released per fission:

$$\begin{aligned} Q &= (\text{Total } B \text{ of products}) - (\text{Total } B \text{ of reactants}) && (7.6)\\ &\approx 235 \times (8.4 - 7.6) && (7.7)\\ &\approx 235 \times 0.8 && (7.8)\\ &\approx 188 \text{ MeV} && (7.9) \end{aligned}$$

This is close to the observed ∼200 MeV.
More precisely, using atomic masses:

$$\begin{aligned} Q &= [M(^{235}\text{U}) + M(n) - M(\text{FF}_1) - M(\text{FF}_2) - \sum M(n_i)]c^2 && (7.10)\\ &\approx 0.215 \text{ u} \times 931.5 \text{ MeV/u} && (7.11)\\ &\approx 200 \text{ MeV} && (7.12) \end{aligned}$$

Comparison with Chemical Energy

The energy density of nuclear fission is extraordinary:

Example

Energy from 1 kg of ^{235}U:
Number of ^{235}U atoms in 1 kg:

$$N = \frac{1000 \text{ g}}{235 \text{ g/mol}} \times 6.022 \times 10^{23} = 2.56 \times 10^{24} \text{ atoms} \quad (7.13)$$

Energy released (assuming all atoms fission):

$$\begin{aligned} E &= N \times 200 \text{ MeV} && (7.14)\\ &= 2.56 \times 10^{24} \times 200 \times 1.602 \times 10^{-13} \text{ J} && (7.15)\\ &= 8.2 \times 10^{13} \text{ J} && (7.16)\\ &\approx 82 \text{ TJ (terajoules)} && (7.17) \end{aligned}$$

Comparison:

- 1 kg ^{235}U ≈ 82 TJ
- 1 kg coal ≈ 30 MJ
- 1 kg gasoline ≈ 45 MJ

Therefore: **1 kg of ^{235}U ≈ 2.7 million kg of coal!**

7.2.4 Fission Fragment Distribution

Fission is predominantly **asymmetric** for thermal neutron-induced fission of ^{235}U.

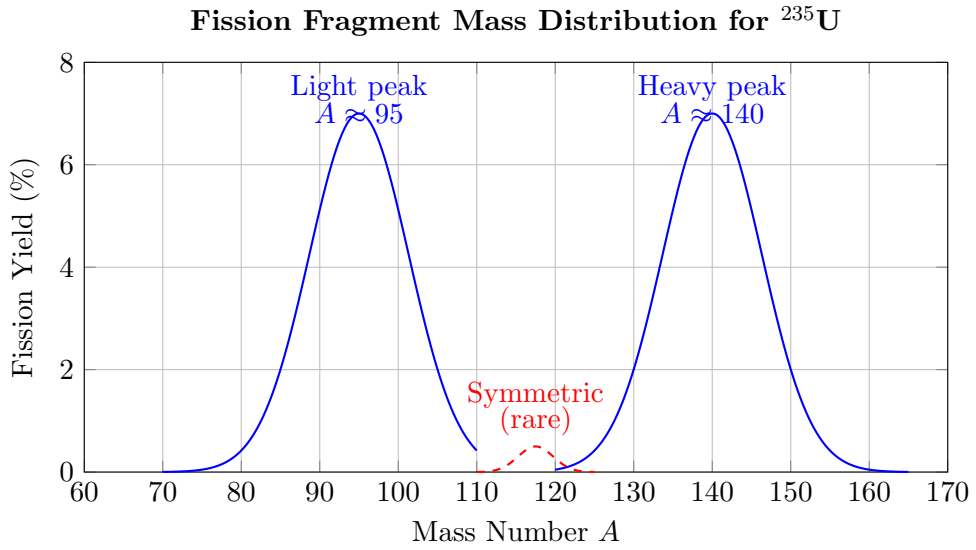


Figure 7.2: Mass distribution of fission fragments from thermal neutron fission of ^{235}U. The distribution shows two peaks (asymmetric fission) with a small valley in the middle (symmetric fission).

Characteristics:

- Light fragment peak: $A \approx 90 - 100$ (e.g., Kr, Rb, Sr)
- Heavy fragment peak: $A \approx 135 - 145$ (e.g., Xe, Cs, Ba)
- Symmetric fission ($A \approx 117$): $< 1\%$ probability
- Asymmetry related to nuclear shell effects

7.2.5 Neutron Emission in Fission

Prompt Neutrons

Fission fragments are neutron-rich (high N/Z ratio) and immediately emit **prompt neutrons** ($< 10^{-14}$ s after fission).

Average number of prompt neutrons per fission:

- ^{235}U: $\bar{\nu} = 2.42$
- ^{239}Pu: $\bar{\nu} = 2.87$
- ^{233}U: $\bar{\nu} = 2.49$

The neutron multiplicity distribution $P(\nu)$ gives the probability of emitting ν neutrons:

Table 7.2: Neutron Multiplicity for ^{235}U Thermal Fission

Number of neutrons (ν)	**Probability $P(\nu)$ (%)**
0	0.3
1	16.6
2	33.6
3	30.2
4	15.1
5	3.7
6	0.5
Average	$\bar{\nu} = 2.42$

Energy spectrum of prompt neutrons:
Approximated by a Maxwellian distribution:

$$N(E) \propto \sqrt{E} \exp(-E/T) \tag{7.18}$$

with most probable energy around 0.7-0.8 MeV and average energy ∼2 MeV.

Delayed Neutrons

A small fraction ($< 1\%$) of neutrons are emitted with delay times from milliseconds to minutes. These **delayed neutrons** come from highly excited fission products that undergo beta decay to states above the neutron separation energy.

Important Formula

Delayed Neutron Fraction:

$$\beta = \frac{\text{Number of delayed neutrons}}{\text{Total number of neutrons}} \tag{7.19}$$

For ^{235}U: $\beta \approx 0.0065$ (0.65%)
For ^{239}Pu: $\beta \approx 0.0021$ (0.21%)

Importance: Despite their small fraction, delayed neutrons are **crucial for reactor control**. Without them, reactor control would be practically impossible because prompt neutrons alone would cause the reactor power to change too rapidly.

Delayed neutron groups:
Delayed neutrons are classified into 6 groups based on their half-lives:

Table 7.3: Delayed Neutron Groups for ^{235}U

Group	**Half-life (s)**	**Decay constant (s^{-1})**	**Yield (%)**
1	55.72	0.0124	0.021
2	22.72	0.0305	0.142
3	6.22	0.111	0.128
4	2.30	0.301	0.257
5	0.614	1.14	0.075
6	0.230	3.01	0.027
	Total		**0.650**

7.3 Chain Reactions and Critical Mass

7.3.1 The Chain Reaction Concept

A **chain reaction** occurs when neutrons produced by one fission event cause additional fissions, which in turn produce more neutrons, and so on.

Definition

Nuclear Chain Reaction: A self-sustaining sequence of fission reactions where neutrons from one fission induce subsequent fissions, potentially leading to an exponential increase in the reaction rate.

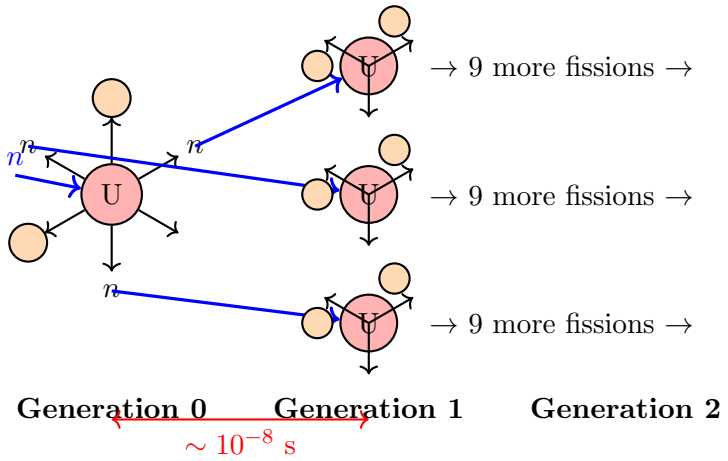


Figure 7.3: Illustration of a nuclear chain reaction. One neutron causes fission, producing 3 neutrons. If all three cause subsequent fissions, the reaction grows exponentially.

7.3.2 Multiplication Factor

The behavior of a chain reaction is characterized by the **multiplication factor** k:

Definition

Effective Multiplication Factor (k or k_{eff}): The ratio of the number of neutrons in one generation to the number in the previous generation.

$$k = \frac{\text{Number of neutrons in generation } n+1}{\text{Number of neutrons in generation } n} \tag{7.20}$$

Three regimes:

1. **Subcritical ($k < 1$):** Chain reaction dies out

$$N(t) = N_0 k^t \to 0 \text{ as } t \to \infty \tag{7.21}$$

2. **Critical ($k = 1$):** Chain reaction is self-sustaining at constant rate

$$N(t) = N_0 \text{ (constant)} \tag{7.22}$$

3. **Supercritical ($k > 1$):** Chain reaction grows exponentially

$$N(t) = N_0 k^t \to \infty \text{ as } t \to \infty \tag{7.23}$$

For reactor operation:

- Power reactors operate at $k = 1$ (critical)

- Nuclear weapons require $k \gg 1$ for rapid energy release

7.3.3 The Four-Factor Formula

The multiplication factor can be expressed as a product of four factors describing different stages of the neutron life cycle:

Important Formula

Four-Factor Formula:

$$k_\infty = \eta \cdot f \cdot p \cdot \varepsilon \tag{7.24}$$

where:

- η = **Reproduction factor**: Average number of fission neutrons produced per neutron absorbed in fuel
- f = **Thermal utilization factor**: Fraction of thermal neutrons absorbed in fuel (vs. other materials)
- p = **Resonance escape probability**: Probability that a neutron slowing down avoids resonance capture
- ε = **Fast fission factor**: Ratio of total fissions to thermal fissions ($\varepsilon > 1$ because some fissions occur with fast neutrons)

This is the **infinite multiplication factor** (for an infinite reactor with no leakage).

For a finite reactor:

$$k_{eff} = k_\infty \cdot P_{NL} \tag{7.25}$$

where P_{NL} is the **non-leakage probability** (fraction of neutrons that don't escape the reactor).

Mathematical Derivation

Calculation of η for ^{235}U:
The reproduction factor is:

$$\eta = \frac{\nu \sigma_f}{\sigma_a} \tag{7.26}$$

where:

- ν = average number of neutrons per fission
- σ_f = fission cross-section
- $\sigma_a = \sigma_f + \sigma_c$ = absorption cross-section (fission + capture)

For thermal neutrons on ^{235}U:

- $\nu = 2.42$
- $\sigma_f = 583$ barns
- $\sigma_c = 99$ barns (radiative capture)
- $\sigma_a = 682$ barns

Therefore:

$$\eta = \frac{2.42 \times 583}{682} = 2.07 \tag{7.27}$$

This means each thermal neutron absorbed in ^{235}U produces on average 2.07 new fission neutrons.

7.3.4 Critical Mass

Definition

Critical Mass: The minimum amount of fissile material needed to maintain a self-sustaining chain reaction ($k_{eff} = 1$).

The critical mass depends on:

- **Material properties:** Cross-sections, ν
- **Geometry:** Sphere minimizes surface area (optimal)
- **Density:** Higher density → smaller critical mass
- **Purity:** Impurities absorb neutrons
- **Neutron reflector:** Reduces leakage, decreases critical mass
- **Moderator:** Slows neutrons for thermal reactors

Approximate critical masses (bare sphere):

- ^{235}U (pure): ∼52 kg (sphere radius ∼8.7 cm)
- ^{239}Pu (pure): ∼10 kg (sphere radius ∼5.0 cm)
- ^{233}U (pure): ∼16 kg

With a good neutron reflector (e.g., beryllium), critical mass can be reduced by factor of 2-3.

Mathematical Derivation

Simple Estimate of Critical Radius:
For a bare sphere of radius R, neutrons can leak out. The critical condition is:

$$k_\infty P_{NL} = 1 \tag{7.28}$$

The non-leakage probability can be approximated using diffusion theory:

$$P_{NL} = \frac{1}{1 + L^2 B^2} \tag{7.29}$$

where L is the diffusion length and B is the geometric buckling.
For a sphere:

$$B^2 = \left(\frac{\pi}{R}\right)^2 \tag{7.30}$$

The critical radius is found when:

$$k_\infty = 1 + L^2 \left(\frac{\pi}{R_c}\right)^2 \tag{7.31}$$

Solving for R_c:

$$R_c = \pi L \sqrt{\frac{1}{k_\infty - 1}} \tag{7.32}$$

For ^{235}U, with $k_\infty \approx 1.3$ and $L \approx 5$ cm:

$$R_c \approx \pi \times 5 \times \sqrt{\frac{1}{0.3}} \approx 29 \text{ cm} \tag{7.33}$$

This is an overestimate because it assumes thermal neutrons; actual critical radius for fast neutrons is smaller (∼8.7 cm).

7.3.5 Reactivity

The deviation from criticality is quantified by **reactivity**:

$$\rho = \frac{k - 1}{k} = \frac{\Delta k}{k} \tag{7.34}$$

Interpretation:

- $\rho > 0$: Supercritical (reactor power increasing)
- $\rho = 0$: Critical (reactor power constant)
- $\rho < 0$: Subcritical (reactor power decreasing)

Units of reactivity:

- Often expressed in "dollars" where 1 dollar = β (delayed neutron fraction)
- For ^{235}U: 1 dollar ≈ 0.0065 in $\Delta k/k$
- Also expressed in "cents": 1 dollar = 100 cents

- pcm (per cent mille): parts per hundred thousand

Prompt critical: When $\rho = \beta$, the reactor is critical on prompt neutrons alone. This is the boundary between controlled and uncontrolled operation.

7.4 Nuclear Reactors

7.4.1 Basic Components of a Nuclear Reactor

A nuclear reactor is a device designed to initiate and control a sustained nuclear chain reaction. All power reactors share common essential components:

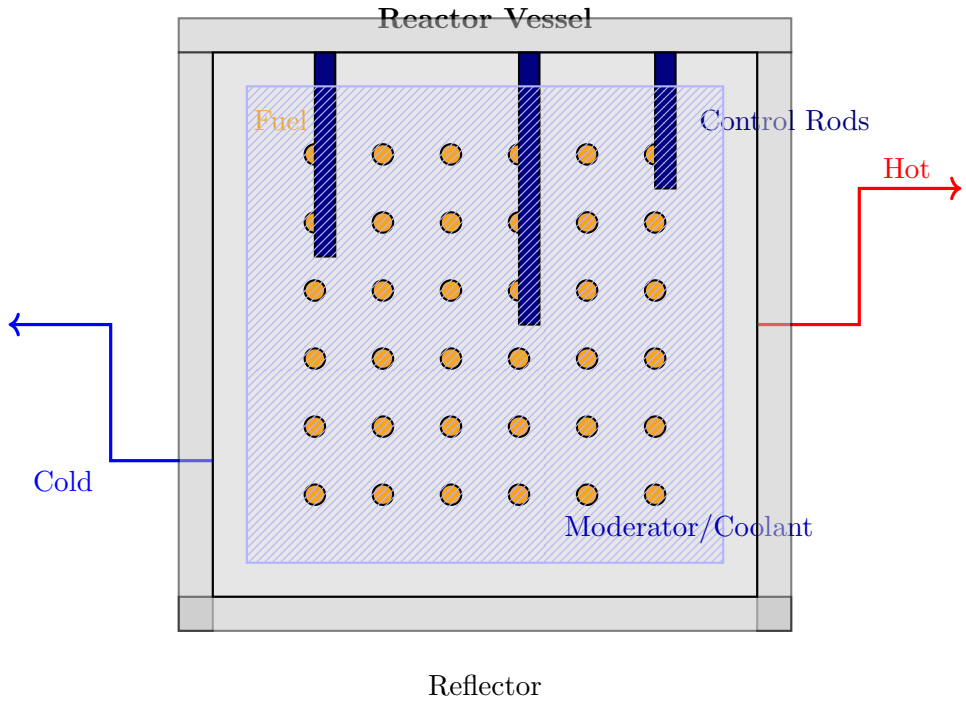


Figure 7.4: Schematic cross-section of a nuclear reactor core showing the main components: fuel rods, control rods, moderator/coolant, and reflector.

1. Nuclear Fuel

The fuel contains fissile material that undergoes fission.

Common fuel types:

- **Uranium oxide (UO_2):** Most common, enriched to 3-5% ^{235}U
- **Mixed oxide (MOX):** Mixture of UO_2 and PuO_2
- **Metallic uranium:** Used in some research reactors

Fuel enrichment:

- Natural uranium: 0.72% ^{235}U, 99.27% ^{238}U
- Low enriched uranium (LEU): 3-5% ^{235}U (power reactors)
- Highly enriched uranium (HEU): $> 20\%$ ^{235}U (research reactors, weapons)

2. Moderator

The moderator slows down fast neutrons to thermal energies where ^{235}U has a much larger fission cross-section.

Good moderator properties:

- Low mass number (efficient energy loss per collision)
- Low neutron absorption cross-section
- High scattering cross-section

Average logarithmic energy decrement:

$$\xi = 1 - \frac{(A-1)^2}{2A} \ln\left(\frac{A+1}{A-1}\right) \tag{7.35}$$

Common moderators:

Table 7.4: Properties of Common Moderators

Material	A	ξ	σ_s **(barns)**	σ_a **(barns)**
H_2O (light water)	1 (H)	0.920	38	0.66
D_2O (heavy water)	2 (D)	0.509	10.6	0.0013
Graphite (C)	12	0.158	4.8	0.0034
Beryllium	9	0.207	6.0	0.010

Number of collisions to thermalize:

$$n = \frac{\ln(E_0/E_{\text{thermal}})}{\xi} \tag{7.36}$$

For 2 MeV neutron to 0.025 eV:

- H_2O: $n \approx 20$ collisions
- D_2O: $n \approx 36$ collisions
- Graphite: $n \approx 115$ collisions

3. Control Rods

Control rods absorb neutrons to control the reaction rate.

Materials with high absorption cross-section:

- Boron-10: $\sigma_a = 3840$ barns
- Cadmium: $\sigma_a = 2450$ barns
- Hafnium: $\sigma_a = 105$ barns

Functions:

- Startup and shutdown
- Power level control
- Emergency shutdown (SCRAM)

Control mechanisms:

- **Position control:** Insertion depth determines absorption
- **Chemical shim:** Soluble neutron absorber in coolant (boric acid)
- **Burnable poisons:** Gradually consumed (e.g., gadolinium in fuel)

4. Coolant

The coolant removes heat from the reactor core.

Requirements:

- High heat capacity
- Good heat transfer properties
- Low neutron absorption (for reactor coolants)
- Chemical stability
- Low corrosiveness

Common coolants:

- **Light water (H_2O):** Acts as both moderator and coolant
- **Heavy water (D_2O):** Excellent moderator, low absorption
- **Liquid sodium:** Fast reactors, excellent heat transfer
- **Helium gas:** High-temperature gas reactors
- **Molten salt:** Advanced reactor designs

5. Reflector

The reflector surrounds the core to reduce neutron leakage.

Materials:

- Graphite
- Beryllium
- Heavy water
- Steel

Benefits:

- Reflects escaped neutrons back into core
- Reduces critical mass
- Flattens power distribution

6. Shielding

Biological shielding protects personnel from radiation.

Radiation types to shield:

- Gamma rays: High-Z materials (lead, steel, concrete)
- Neutrons: Hydrogenous materials (water, concrete)

Typical shield: 2-3 meters of concrete

7.4.2 Types of Nuclear Reactors

Classification by Neutron Energy

1. Thermal Reactors
Fission occurs primarily with thermal neutrons (energy ∼0.025 eV).
Characteristics:

- Requires moderator
- Can use natural or low-enriched uranium (with good moderator)
- Most common type
- Examples: PWR, BWR, CANDU, RBMK

Advantages:

- Large fission cross-sections at thermal energies
- Well-established technology
- Can use LEU fuel

Disadvantages:

- Less efficient neutron economy
- Cannot breed effectively

2. Fast Reactors
Fission occurs with fast neutrons (energy ∼0.1-1 MeV).
Characteristics:

- No moderator
- Requires enriched fuel or plutonium
- Higher power density
- Can breed fissile material
- Examples: BN-800, Phenix, EBR-II

Advantages:

- Can breed more fissile material than consumed
- Better fuel utilization
- Can burn actinides and long-lived waste

Disadvantages:

- More complex design
- Requires liquid metal coolant (typically sodium)
- Less operational experience

Pressurized Water Reactor (PWR)

Most common reactor type worldwide (∼60% of operating reactors).

Design features:

- Light water moderator and coolant
- Primary loop at high pressure (155 bar)
- Water doesn't boil in core
- Secondary steam loop for turbine
- Fuel: UO_2 enriched to 3-5% ^{235}U

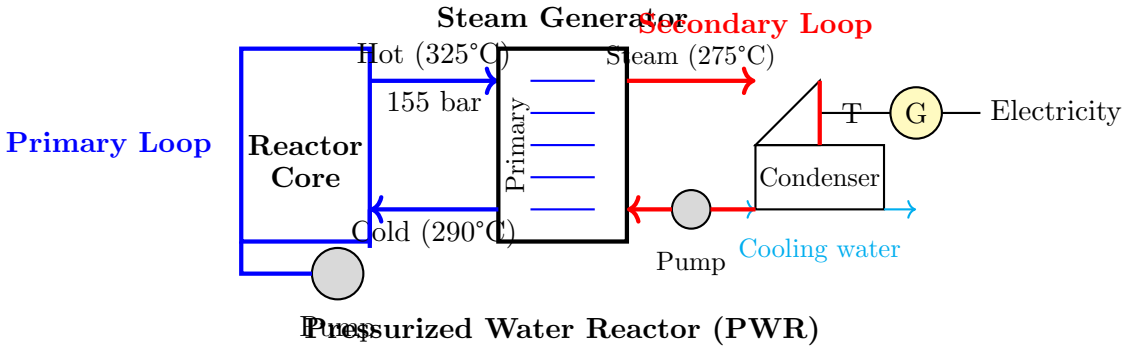


Figure 7.5: Schematic diagram of a Pressurized Water Reactor (PWR) showing the primary loop (radioactive) and secondary loop (non-radioactive) separated by the steam generator.

Advantages:

- Proven technology (hundreds in operation)
- Compact design
- Good safety features (negative temperature coefficient)
- Radioactive water confined to primary loop

Disadvantages:

- High pressure system
- Requires enriched fuel
- Large containment structure needed

Boiling Water Reactor (BWR)

Second most common type (∼20% of operating reactors).

Design features:

- Light water moderator and coolant
- Water boils in reactor core
- Direct cycle (steam goes directly to turbine)

- Lower pressure than PWR (75 bar)

Advantages:

- Simpler design (no steam generator)
- Lower pressure than PWR
- Better thermal efficiency

Disadvantages:

- Radioactive steam in turbine building
- More complex control (void reactivity)
- Turbine contamination

CANDU Reactor

CANada Deuterium Uranium reactor - distinctive Canadian design.

Design features:

- Heavy water (D_2O) moderator and coolant
- Natural uranium fuel (no enrichment needed)
- Horizontal pressure tubes (not pressure vessel)
- On-line refueling capability

Advantages:

- Can use natural uranium fuel
- Excellent neutron economy
- Can refuel without shutdown
- Flexible fuel options

Disadvantages:

- Expensive heavy water
- Large size
- Produces more plutonium (proliferation concern)

Breeder Reactors

Breeder reactors produce more fissile material than they consume.

Definition

Breeding: The process of converting fertile material (e.g., ^{238}U or ^{232}Th) into fissile material (e.g., ^{239}Pu or ^{233}U).

Breeding reactions:

1. Uranium-Plutonium cycle:

$$^{238}\text{U} + n \rightarrow ^{239}\text{U} \tag{7.37}$$

$$^{239}\text{U} \rightarrow ^{239}\text{Np} + \beta^- \quad (t_{1/2} = 23.5 \text{ min}) \tag{7.38}$$

$$^{239}\text{Np} \rightarrow ^{239}\text{Pu} + \beta^- \quad (t_{1/2} = 2.36 \text{ days}) \tag{7.39}$$

2. Thorium-Uranium cycle:

$$^{232}\text{Th} + n \rightarrow ^{233}\text{Th} \tag{7.40}$$

$$^{233}\text{Th} \rightarrow ^{233}\text{Pa} + \beta^- \quad (t_{1/2} = 22 \text{ min}) \tag{7.41}$$

$$^{233}\text{Pa} \rightarrow ^{233}\text{U} + \beta^- \quad (t_{1/2} = 27 \text{ days}) \tag{7.42}$$

Breeding ratio:

$$BR = \frac{\text{Rate of fissile material production}}{\text{Rate of fissile material consumption}} \tag{7.43}$$

For breeding: $BR > 1$

Conversion ratio:

$$CR = \frac{\text{Fissile atoms produced}}{\text{Fissile atoms consumed}} \tag{7.44}$$

- Thermal reactors: $CR \approx 0.6$ (convert but don't breed)
- Fast breeder reactors: $BR \approx 1.1 - 1.4$ (actually breed)

Advantages of breeding:

- Extends uranium resources by factor of 50-60
- Can utilize depleted uranium stockpiles
- Better fuel utilization

Examples:

- BN-800 (Russia) - operating
- Phénix (France) - shut down
- Monju (Japan) - shut down
- EBR-II (USA) - shut down

7.4.3 Reactor Control and Safety

Reactivity Coefficients

Reactors have inherent feedback mechanisms that affect reactivity:

1. Temperature coefficient:

$$\alpha_T = \frac{d\rho}{dT} \tag{7.45}$$

Negative temperature coefficient (desirable for safety):

- Temperature increases → reactivity decreases → power decreases
- Provides automatic stabilization

2. Void coefficient:

$$\alpha_V = \frac{d\rho}{d(\text{void fraction})} \tag{7.46}$$

In water-moderated reactors:

- Positive void coefficient: dangerous (if coolant boils, reactivity increases)
- Negative void coefficient: safe (if coolant boils, reactivity decreases)

3. Power coefficient:

$$\alpha_P = \frac{d\rho}{dP} \tag{7.47}$$

Combines temperature and void effects.

Reactor Kinetics

The time evolution of neutron population is described by the **point kinetics equations**:
For prompt neutrons:

$$\frac{dn}{dt} = \frac{\rho - \beta}{\Lambda} n + \sum_{i=1}^{6} \lambda_i C_i \tag{7.48}$$

For delayed neutron precursors:

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} n - \lambda_i C_i \tag{7.49}$$

where:

- n = neutron density
- ρ = reactivity
- β = delayed neutron fraction
- Λ = prompt neutron lifetime
- C_i = concentration of i-th precursor group
- λ_i = decay constant of i-th precursor group

Prompt neutron lifetime:

- Thermal reactors: $\Lambda \approx 10^{-4}$ s
- Fast reactors: $\Lambda \approx 10^{-7}$ s

Stable period equation:
For small reactivity insertion:

$$T = \frac{\bar{\ell}}{\rho - \beta} \tag{7.50}$$

where T is the reactor period and $\bar{\ell}$ is the effective neutron generation time.

Safety Systems

1. Control systems:

- Automatic control rods
- Chemical shim adjustment
- Power distribution monitoring

2. Emergency shutdown (SCRAM):

- Rapid insertion of all control rods
- Gravity-driven (fail-safe)
- Can be triggered by multiple parameters

3. Passive safety features:

- Negative temperature coefficient
- Natural circulation cooling
- Containment structure
- Multiple barriers (fuel cladding, primary system, containment)

4. Active safety systems:

- Emergency core cooling systems (ECCS)
- Backup power (diesel generators)
- Pressure relief systems

7.4.4 Controlled vs. Uncontrolled Reactions

Table 7.5: Comparison of Controlled and Uncontrolled Fission Reactions

Characteristic	**Controlled (Reactor)**	**Uncontrolled (Weapon)**
Multiplication factor	$k \approx 1$ (critical)	$k \gg 1$ (highly supercritical)
Neutron energy	Thermal ($\sim$0.025 eV) or fast	Fast ($\sim$1 MeV)
Fuel enrichment	3-5% (LEU) or 20%	$>$ 90% (weapons-grade)
Critical mass	Large (tons), subcritical without moderator	Small (kg), supercritical
Geometry	Distributed, optimized for control	Compact, optimized for yield
Time scale	Months to years	Microseconds
Energy release rate	MW to GW (steady)	$> 10^{15}$ W (explosive)
Purpose	Sustained power production	Rapid energy release
Control mechanism	Control rods, moderator, delayed neutrons	None (uncontrolled)
Delayed neutrons	Essential for control	Minimal role
Efficiency	$\sim$3-5% of fuel fissioned	$<$20% even in best weapons

7.5 Nuclear Fusion

7.5.1 The Fusion Process

Nuclear fusion is the process by which two light nuclei combine to form a heavier nucleus, releasing energy.

Definition

Nuclear Fusion: A nuclear reaction in which two or more light atomic nuclei collide at high energy and fuse together to form a heavier nucleus, releasing energy in the process.

Why fusion releases energy:

Fusion of light nuclei moves them toward the peak of the binding energy curve (near Fe-56), increasing the binding energy per nucleon.

$$Q = [M_1 + M_2 - M_3 - M_4]c^2 > 0 \tag{7.51}$$

for light nuclei ($A < 56$).

7.5.2 The Coulomb Barrier

The main challenge in fusion is overcoming the **Coulomb barrier** - the electrostatic repulsion between positively charged nuclei.

Coulomb potential energy:

$$V_C(r) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} = \frac{1.44 \text{ MeV·fm}}{r} \times Z_1 Z_2 \tag{7.52}$$

At nuclear contact ($r \approx 2$ fm for light nuclei):

$$V_C \approx 0.72 Z_1 Z_2 \text{ MeV} \tag{7.53}$$

For deuterium-tritium fusion:

$$V_C \approx 0.72 \times 1 \times 1 \approx 0.4 \text{ MeV} \tag{7.54}$$

Classical requirement: Nuclei need kinetic energy $E \geq V_C$ to fuse.
Using $E = \frac{3}{2}kT$:

$$T = \frac{2V_C}{3k} \approx \frac{2 \times 0.4 \times 10^6 \times 1.6 \times 10^{-19}}{3 \times 1.38 \times 10^{-23}} \approx 3 \times 10^9 \text{ K} \tag{7.55}$$

Quantum tunneling: Fusion can occur at lower energies due to quantum tunneling through the Coulomb barrier. Significant fusion rates occur at:

$$T \approx 10^7 - 10^8 \text{ K} \tag{7.56}$$

7.5.3 Important Fusion Reactions

Deuterium-Tritium (D-T) Reaction

The most promising reaction for terrestrial fusion:

$$^2\text{H} + ^3\text{H} \rightarrow ^4\text{He}(3.5 \text{ MeV}) + n(14.1 \text{ MeV}) \tag{7.57}$$

Total energy: $Q = 17.6$ MeV
Advantages:

- Highest cross-section at achievable temperatures
- Peak cross-section at ∼64 keV
- Relatively low ignition temperature (∼4 keV ≈ 50 million K)

Disadvantages:

- Tritium is radioactive ($t_{1/2} = 12.3$ years)
- Tritium must be bred from lithium
- High-energy neutrons damage reactor materials

Deuterium-Deuterium (D-D) Reactions

Two competing reactions:

$$^2\text{H} + ^2\text{H} \rightarrow ^3\text{He}(0.82\text{ MeV}) + n(2.45\text{ MeV}) \quad (50\%) \tag{7.58}$$

$$^2\text{H} + ^2\text{H} \rightarrow ^3\text{H}(1.01\text{ MeV}) + p(3.02\text{ MeV}) \quad (50\%) \tag{7.59}$$

Advantages:

- Deuterium is abundant (seawater)
- No radioactive fuel

Disadvantages:

- Lower cross-section than D-T
- Higher ignition temperature (∼10 keV)
- Lower energy release per reaction

Proton-Boron Reaction (Aneutronic)

$$p + ^{11}\text{B} \rightarrow 3\,^4\text{He} + 8.7\text{ MeV} \tag{7.60}$$

Advantages:

- No neutrons (aneutronic)
- No radioactive products
- Direct energy conversion possible

Disadvantages:

- Extremely high temperature required (∼100 keV)
- Very low cross-section
- Technically very challenging

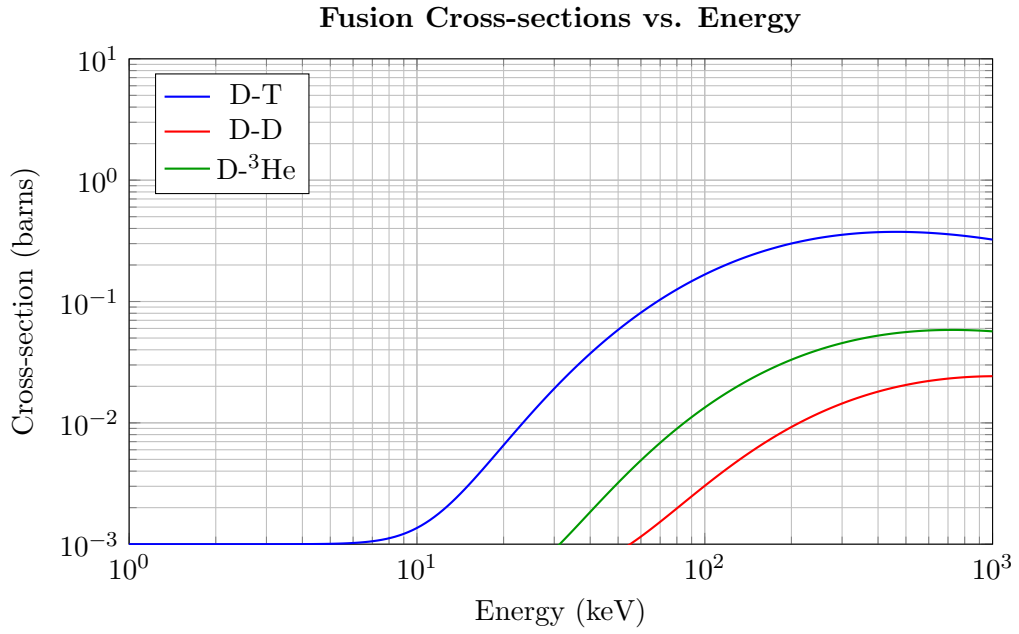


Figure 7.6: Fusion cross-sections for different reactions as a function of energy. D-T has the highest cross-section at the lowest energies, making it the most favorable for practical fusion.

7.5.4 Energy Production in Stars

Stars generate energy through thermonuclear fusion in their cores.

The Proton-Proton Chain

Dominant in stars like our Sun ($T \approx 15 \times 10^6$ K):

PP-I chain (85%):

$$p + p \rightarrow {}^2\text{H} + e^+ + \nu_e \quad (Q = 1.44 \text{ MeV}) \tag{7.61}$$

$$^2\text{H} + p \rightarrow {}^3\text{He} + \gamma \quad (Q = 5.49 \text{ MeV}) \tag{7.62}$$

$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2p \quad (Q = 12.86 \text{ MeV}) \tag{7.63}$$

Overall reaction:

$$4p \rightarrow {}^4\text{He} + 2e^+ + 2\nu_e + 2\gamma + 26.7 \text{ MeV} \tag{7.64}$$

(Note: $2e^+ + 2e^- \rightarrow 4\gamma$ adds 2.04 MeV, total = 28.3 MeV, but neutrinos carry away ~0.5 MeV)

Net energy available: ~26.2 MeV per helium nucleus created

The CNO Cycle

Dominant in more massive stars ($T > 18 \times 10^6$ K):

$$^{12}\text{C} + p \rightarrow {}^{13}\text{N} + \gamma \tag{7.65}$$

$$^{13}\text{N} \rightarrow {}^{13}\text{C} + e^+ + \nu_e \tag{7.66}$$

$$^{13}\text{C} + p \rightarrow {}^{14}\text{N} + \gamma \tag{7.67}$$

$$^{14}\text{N} + p \rightarrow {}^{15}\text{O} + \gamma \tag{7.68}$$

$$^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + \nu_e \tag{7.69}$$

$$^{15}\text{N} + p \rightarrow {}^{12}\text{C} + {}^4\text{He} \tag{7.70}$$

Net reaction:

$$4p \rightarrow {}^{4}\text{He} + 2e^{+} + 2\nu_e + 3\gamma + 25.0 \text{ MeV} \tag{7.71}$$

Carbon acts as a catalyst.

Solar Energy Production

Example

Solar luminosity from fusion:
The Sun's luminosity: $L_\odot = 3.828 \times 10^{26}$ W
Energy per fusion: $Q = 26.2$ MeV $= 4.2 \times 10^{-12}$ J
Fusion rate:

$$R = \frac{L_\odot}{Q} = \frac{3.828 \times 10^{26}}{4.2 \times 10^{-12}} = 9.1 \times 10^{37} \text{ fusions/s} \tag{7.72}$$

Mass converted to energy per second:

$$\frac{dm}{dt} = \frac{L_\odot}{c^2} = \frac{3.828 \times 10^{26}}{9 \times 10^{16}} = 4.3 \times 10^{9} \text{ kg/s} \tag{7.73}$$

The Sun converts 4.3 million tons of mass to energy every second!
Over its 10-billion-year lifetime, the Sun will convert only ∼0.07% of its mass to energy.

7.5.5 Controlled Thermonuclear Fusion

The Lawson Criterion

For a fusion reactor to be viable, it must achieve **net energy gain**. The Lawson criterion specifies the minimum conditions:

Important Formula

Lawson Criterion:

$$n\tau > \frac{12kT}{\sigma v E_{\text{fusion}}} \tag{7.74}$$

where:

- n = plasma density (particles/m^3)
- τ = confinement time (s)
- T = temperature
- σv = reactivity
- E_{fusion} = energy per fusion

For D-T fusion at $T = 10$ keV:

$$n\tau > 10^{20} \text{ m}^{-3} \cdot \text{s} \tag{7.75}$$

The "triple product":

$$nT\tau > 5 \times 10^{21} \text{ m}^{-3} \cdot \text{keV} \cdot \text{s} \tag{7.76}$$

This is often used as a figure of merit for fusion experiments.
Breakeven: Energy out = Energy in

Ignition: Fusion-produced alpha particles heat the plasma sufficiently to sustain the reaction without external heating.

Magnetic Confinement Fusion

Uses strong magnetic fields to confine hot plasma.

Tokamak (most developed):

- Toroidal (doughnut-shaped) chamber
- Combination of toroidal and poloidal magnetic fields
- Plasma temperature: 100-200 million K
- Density: 10^{20} m^{-3}
- Confinement time: several seconds

ITER (International Thermonuclear Experimental Reactor):

- Major radius: 6.2 m
- Plasma volume: 840 m^3
- Magnetic field: 5.3 T
- Target: $Q = 10$ (10× energy gain)
- Expected first plasma: 2025

Stellarator:

- Twisted toroidal design
- No need for plasma current
- Steady-state operation
- More complex engineering

Inertial Confinement Fusion (ICF)

Uses intense energy beams to compress and heat a fuel pellet.

Approach:

1. Fuel pellet (D-T ice, ∼2 mm diameter)
2. Intense laser/ion beams heat outer layer
3. Outer layer ablates, compressing core
4. Core reaches high density and temperature
5. Fusion ignition occurs

National Ignition Facility (NIF):

- 192 laser beams
- 1.8 MJ energy delivered

- Compression to 100× liquid density
- Temperature: 100 million K
- In 2022: First achievement of net energy gain!

Recent breakthrough (December 2022):

- Input energy: 2.05 MJ
- Output energy: 3.15 MJ
- Energy gain: 1.5×
- First controlled fusion with net energy output

7.5.6 The Hydrogen Bomb

An uncontrolled fusion weapon using fission to trigger fusion.

Teller-Ulam design:

1. **Primary stage (fission):**
   - Plutonium or uranium fission bomb
   - Produces X-rays and high temperature
2. **Secondary stage (fusion):**
   - X-rays compress fusion fuel (LiD)
   - Fusion of deuterium and tritium
   - Lithium deuteride: $^{6}\text{Li} + \text{n} \rightarrow {}^{3}\text{H} + {}^{4}\text{He}$
   - Tritium fuses with deuterium
3. **Tertiary effects:**
   - 14 MeV neutrons can fission U-238 (additional yield)
   - Multiple stages possible

Energy yield:

- Fission weapons: <1 megaton TNT equivalent
- Fusion weapons: tens of megatons possible
- Largest ever: Tsar Bomba (50 Mt, 1961)

Difference from reactor:

- Microsecond time scale
- No attempt at control
- Extreme temperatures and pressures
- Designed for maximum energy release

7.6 Summary

In this comprehensive chapter, we have explored the two fundamental processes that release nuclear energy:

7.6.1 Nuclear Fission

- **Mechanism:** Heavy nuclei split into lighter fragments, releasing ∼200 MeV per fission
- **Chain reactions:** Sustained by neutrons from fission; controlled by multiplication factor k
- **Critical mass:** Minimum mass needed for sustained chain reaction
- **Delayed neutrons:** Crucial for reactor control despite being <1% of total
- **Reactors:** Convert nuclear energy to electricity through various designs (PWR, BWR, CANDU, fast breeder)
- **Control:** Achieved through control rods, moderators, and inherent feedback mechanisms

7.6.2 Nuclear Fusion

- **Mechanism:** Light nuclei combine to form heavier nuclei, releasing energy
- **Coulomb barrier:** Main challenge requiring extreme temperatures ($> 10^7$ K)
- **Stellar fusion:** Powers stars through PP chain and CNO cycle
- **D-T reaction:** Most promising for terrestrial fusion (17.6 MeV per fusion)
- **Lawson criterion:** Defines conditions for net energy gain: $n\tau > 10^{20}$ m^{-3}·s
- **Confinement:** Magnetic (tokamaks) or inertial (laser compression)
- **Recent progress:** NIF achieved net energy gain in 2022

Key Point

Key Takeaways:

- Both fission and fusion move nuclei toward maximum binding energy (Fe-56 region)
- Fission is a mature technology providing ∼10% of world electricity
- Fusion promises abundant, clean energy but remains technically challenging
- Delayed neutrons enable reactor control; without them, control would be impossible
- Breeding extends nuclear fuel resources by factor of 50-60
- Controlled fusion requires satisfying the Lawson criterion
- Recent fusion breakthrough (NIF) demonstrates feasibility of net energy gain

7.7 Practice Problems

1. Calculate the energy released in the fission reaction:

$$n + {}^{235}\text{U} \rightarrow {}^{92}\text{Kr} + {}^{141}\text{Ba} + 3n$$

   Given: M(U-235) = 235.044 u, M(Kr-92) = 91.926 u, M(Ba-141) = 140.914 u, $M(n)$ = 1.009 u

2. A reactor operates at 1000 MW thermal power. How many fissions occur per second? How much ^{235}U is consumed per day?

3. For ^{235}U, given $\nu = 2.42$, what fraction of neutrons must cause fission (rather than being absorbed or escaping) for $k = 1$?

4. Calculate the number of collisions required to thermalize a 2 MeV neutron to 0.025 eV in (a) H_2O, (b) D_2O, (c) graphite.

5. Estimate the temperature required for D-T fusion assuming classical physics (no tunneling). Compare this to the actual operating temperature of fusion reactors.

6. For the proton-proton chain in the Sun, calculate the mass difference and verify the energy release of 26.7 MeV.

7. A tokamak achieves $n = 10^{20}$ m^{-3} and $T = 10$ keV. What confinement time is needed to satisfy the Lawson criterion for D-T fusion?

8. Explain why delayed neutrons, despite being less than 1% of total neutrons, are essential for reactor control. What would happen if there were no delayed neutrons?

9. Calculate the breeding ratio for a reactor where each fission produces 2.5 neutrons, 1.0 causes the next fission, 0.3 are absorbed in structural materials, 0.2 escape, and the remainder convert fertile to fissile material.

10. Compare the energy density (J/kg) of: (a) coal, (b) gasoline, (c) ^{235}U fission, (d) D-T fusion. What are the implications for fuel requirements?

End of Chapter 7: Nuclear Fission and Fusion

Chapter 8

Particle Accelerators and Detectors

8.1 Introduction

The study of nuclear and particle physics relies heavily on our ability to probe the structure of matter at the smallest scales. This requires two essential tools: particle accelerators to provide high-energy particles for collisions and experiments, and detectors to observe and measure the products of these interactions. Together, these instruments have revolutionized our understanding of the atomic nucleus, fundamental particles, and the forces that govern them.

In this chapter, we will explore the principles behind various types of particle accelerators and the detection methods used to observe nuclear and subatomic particles. These instruments represent some of humanity's most sophisticated technological achievements and continue to push the boundaries of our knowledge.

8.2 The Need for Particle Accelerators

8.2.1 Why Accelerate Particles?

Particle accelerators serve several crucial purposes in nuclear and particle physics:

1. Probing Nuclear Structure

To investigate the internal structure of nuclei and nucleons, we need particles with wavelengths comparable to or smaller than nuclear dimensions. The de Broglie wavelength is:

$$\lambda = \frac{h}{p} \tag{8.1}$$

For nuclear dimensions ($\sim 10^{-15}$ m), we need:

$$p = \frac{h}{\lambda} \approx \frac{6.63 \times 10^{-34}}{10^{-15}} \approx 6.63 \times 10^{-19} \text{ kg·m/s} \tag{8.2}$$

The corresponding kinetic energy is:

$$E \approx \frac{p^2}{2m} \approx 1.2 \text{ MeV (for protons)} \tag{8.3}$$

For subnuclear structure (quarks), we need energies in the GeV to TeV range.

2. Overcoming the Coulomb Barrier

For nuclear reactions involving charged particles, we must overcome the Coulomb repulsion:

$$V_C = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} \tag{8.4}$$

For typical nuclear reactions, this requires particle energies of several MeV or more.

3. Creating New Particles

According to Einstein's mass-energy equivalence, we can create particles if we provide sufficient energy:

$$E = mc^2 \tag{8.5}$$

Examples:

- Proton-antiproton pair: $E \geq 2m_p c^2 \approx 1.88$ GeV
- W boson: $E \geq m_W c^2 \approx 80$ GeV
- Higgs boson: $E \geq m_H c^2 \approx 125$ GeV

4. Medical and Industrial Applications

- **Cancer therapy:** Proton and heavy ion beams
- **Isotope production:** Medical radioisotopes
- **Materials modification:** Ion implantation
- **Sterilization:** Electron beam irradiation

8.2.2 Energy Units in Accelerators

In accelerator physics, we commonly use the **electron volt (eV)** as the unit of energy:

Definition

Electron Volt: The kinetic energy gained by an electron when accelerated through a potential difference of 1 volt.

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \tag{8.6}$$

Common multiples:

- 1 keV $= 10^3$ eV
- 1 MeV $= 10^6$ eV
- 1 GeV $= 10^9$ eV
- 1 TeV $= 10^{12}$ eV

8.3 Principles of Particle Acceleration

8.3.1 Basic Acceleration Mechanism

A charged particle with charge q in an electric field $\vec{E}$ experiences a force:

$$\vec{F} = q\vec{E} \tag{8.7}$$

When accelerated through a potential difference V, the particle gains kinetic energy:

$$\Delta E = qV \tag{8.8}$$

For an electron ($q = -e$) or proton ($q = +e$) accelerated through 1 MV:

$$\Delta E = 1 \text{ MeV} \tag{8.9}$$

8.3.2 Non-Relativistic vs. Relativistic Regime

Non-relativistic ($v \ll c$):

$$E_k = \frac{1}{2}mv^2 = qV \tag{8.10}$$

$$v = \sqrt{\frac{2qV}{m}} \tag{8.11}$$

Relativistic ($v \approx c$):

$$E_{total} = \gamma mc^2 = mc^2 + E_k \tag{8.12}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

$$E_k = (\gamma - 1)mc^2 = qV \tag{8.13}$$

Example

Energy and velocity of a 10 MeV proton:
For a proton: $m_p c^2 = 938.3$ MeV
Since $E_k = 10$ MeV $\ll m_p c^2$, we can use non-relativistic approximation:

$$v = \sqrt{\frac{2E_k}{m_p}} = \sqrt{\frac{2 \times 10 \times 1.602 \times 10^{-13}}{1.673 \times 10^{-27}}} = 4.38 \times 10^7 \text{ m/s} \tag{8.14}$$

$$\frac{v}{c} = 0.146 \approx 14.6\% \text{ of speed of light} \tag{8.15}$$

Error check: $\gamma = 1.011$ (1.1% relativistic correction) - acceptable approximation.

8.4 Types of Particle Accelerators

Particle accelerators can be broadly classified into two categories:

1. **Linear accelerators (linacs):** Particles travel in a straight line
2. **Circular accelerators:** Particles travel in circular or spiral paths

8.4.1 Linear Accelerators (Linacs)

Basic Principle

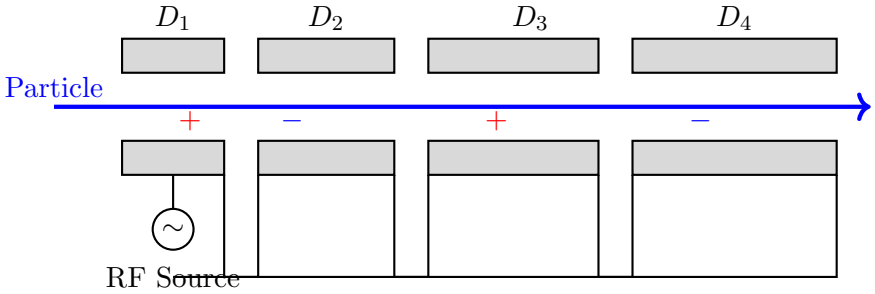
A linear accelerator accelerates particles along a straight path using a series of accelerating structures. The most common type uses radio-frequency (RF) electromagnetic fields.

Key components:

- Particle source (electron gun, ion source)
- Series of drift tubes or RF cavities
- RF power source
- Focusing elements (quadrupole magnets)
- Vacuum system

Wideröe Linear Accelerator

The simplest type of linac, proposed by Rolf Wideröe in 1928.



Wideröe Linear Accelerator

Figure 8.1: Schematic diagram of a Wideröe linear accelerator. Drift tubes alternate in polarity, accelerating the particle in the gaps.

Operating principle:

1. Drift tubes are connected to alternating terminals of an RF source
2. Particles are accelerated in gaps between tubes
3. Inside tubes, particles drift (shielded from field)
4. RF frequency synchronized so particle always sees accelerating field

Drift tube length:
The particle must spend exactly half an RF period inside each tube:

$$L_n = v_n \cdot \frac{T}{2} = \frac{v_n}{2f} \tag{8.16}$$

where v_n is the velocity in the n-th tube and f is the RF frequency.

Mathematical Derivation

Energy and velocity progression:
After n accelerations through voltage V:

$$E_n = nqV \tag{8.17}$$

For non-relativistic particles:

$$\frac{1}{2}mv_n^2 = nqV \tag{8.18}$$

$$v_n = \sqrt{\frac{2nqV}{m}} \tag{8.19}$$

Drift tube length:

$$L_n = \frac{v_n}{2f} = \frac{1}{2f}\sqrt{\frac{2nqV}{m}} \propto \sqrt{n} \tag{8.20}$$

Thus, drift tube lengths increase as $\sqrt{n}$.

Modern RF Linacs

Modern linear accelerators use resonant RF cavities instead of drift tubes.

Advantages:

- Higher accelerating gradients (10-50 MV/m)
- Better efficiency
- Can accelerate electrons to nearly the speed of light

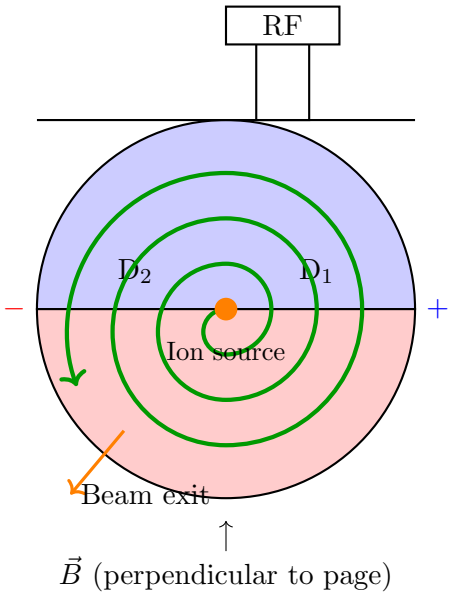
Examples:

- SLAC (Stanford Linear Accelerator): 3 km long, 50 GeV electrons
- European XFEL: 1.7 km, used for X-ray free-electron laser
- Medical linacs: 5-25 MeV electrons for cancer therapy

8.4.2 Cyclotron

The cyclotron, invented by Ernest O. Lawrence in 1929 (Nobel Prize 1939), uses a constant magnetic field and alternating electric field to accelerate particles in a spiral path.

Basic Structure and Operation



Cyclotron (Top View)

Figure 8.2: Schematic diagram of a cyclotron showing the two D-shaped electrodes (dees), spiral particle path, and RF oscillator.

Components:

- Two D-shaped hollow electrodes ("dees")
- Uniform magnetic field perpendicular to dees
- RF voltage source connected to dees
- Ion source at center
- Vacuum chamber

Cyclotron Principle

A charged particle in a magnetic field experiences the Lorentz force:

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{8.21}$$

This provides centripetal force:

$$qvB = \frac{mv^2}{r} \tag{8.22}$$

The radius of circular motion:

Important Formula

Cyclotron Radius:

$$r = \frac{mv}{qB} \tag{8.23}$$

The angular frequency (cyclotron frequency):

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{8.24}$$

The period of revolution:

Important Formula

Cyclotron Frequency and Period:

$$f = \frac{\omega}{2\pi} = \frac{qB}{2\pi m} \tag{8.25}$$

$$T = \frac{1}{f} = \frac{2\pi m}{qB} \tag{8.26}$$

Key insight: The period is *independent of velocity and radius*!

Operation:

1. Ions start at center and are accelerated by electric field in gap
2. Inside dees, ions follow semicircular path (no electric field)
3. RF voltage alternates with frequency $f = qB/(2\pi m)$
4. Each crossing of gap provides energy $\Delta E = qV$
5. Spiral radius increases as energy increases
6. Particles extracted when reaching maximum radius

Mathematical Derivation

Maximum energy of cyclotron:
At maximum radius R:

$$R = \frac{mv_{max}}{qB} \tag{8.27}$$

$$v_{max} = \frac{qBR}{m} \tag{8.28}$$

Maximum kinetic energy (non-relativistic):

$$E_{max} = \frac{1}{2}mv_{max}^2 = \frac{q^2B^2R^2}{2m} \tag{8.29}$$

For a proton in $B = 1.5$ T, $R = 0.5$ m:

$$E_{max} = \frac{(1.602 \times 10^{-19})^2 \times (1.5)^2 \times (0.5)^2}{2 \times 1.673 \times 10^{-27}} \tag{8.30}$$

$$= 1.73 \times 10^{-12} \text{ J} = 10.8 \text{ MeV} \tag{8.31}$$

Relativistic Limitation

As particles approach relativistic speeds, the mass increases:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{8.32}$$

The cyclotron frequency becomes:

$$f = \frac{qB}{2\pi\gamma m_0} \tag{8.33}$$

The frequency *decreases* with energy, causing the particle to fall out of synchronization with the RF field.

Practical limit: Cyclotrons work well for protons up to ∼25 MeV. For higher energies, synchrotrons are needed.

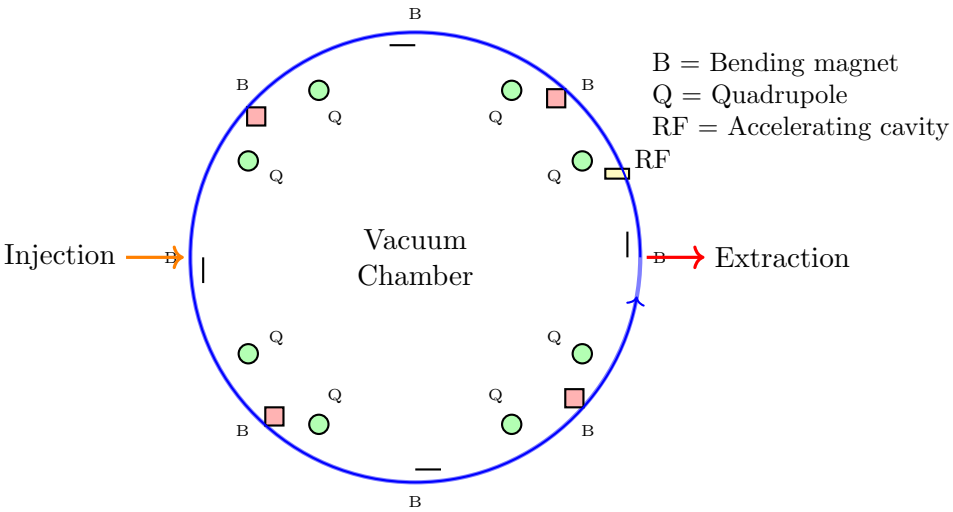
8.4.3 Synchrotron

The synchrotron overcomes the relativistic limitation by varying either the magnetic field or the RF frequency (or both) to keep particles synchronized.

Basic Principle

Key features:

- Particles circulate in a ring of constant radius
- Magnetic field increases as particle energy increases
- RF frequency adjusted to maintain synchronization
- Particles injected at high energy from smaller accelerator



Synchrotron Layout

Figure 8.3: Schematic layout of a synchrotron showing bending magnets, quadrupole focusing magnets, and RF accelerating cavities arranged around a circular ring.

Synchronization Condition

For a particle with momentum p in a circular orbit of radius R:

$$R = \frac{p}{qB} \tag{8.34}$$

For relativistic particles:

$$p = \gamma m_0 v \tag{8.35}$$

To maintain constant R, the magnetic field must increase with momentum:

$$B(t) = \frac{p(t)}{qR} \tag{8.36}$$

The revolution frequency:

$$f_{rev} = \frac{v}{2\pi R} \tag{8.37}$$

For highly relativistic particles ($v \approx c$):

$$f_{rev} \approx \frac{c}{2\pi R} \text{ (approximately constant)} \tag{8.38}$$

Major Synchrotrons

Table 8.1: Major Synchrotron Accelerators

Facility	Location	Circumference (km)	Energy (TeV)
Large Hadron Collider (LHC)	CERN, Switzerland	27	13 (protons)
Tevatron (shut down)	Fermilab, USA	6.3	1.96 (p-$\bar{p}$)
Relativistic Heavy Ion Collider	BNL, USA	3.8	0.25 (Au ions)
Super Proton Synchrotron	CERN, Switzerland	7	0.45

8.4.4 Betatron

The betatron accelerates electrons using a changing magnetic flux, based on Faraday's law of electromagnetic induction.

Operating Principle

Basic idea:

- Electrons orbit in a circular vacuum chamber (donut-shaped)
- Changing magnetic flux through the orbit induces electric field
- Induced electric field accelerates electrons
- Simultaneously, bending magnetic field keeps electrons in orbit

Faraday's law:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{8.39}$$

For circular path of radius r:

$$2\pi r E = -\frac{d\Phi}{dt} \tag{8.40}$$

$$E = -\frac{1}{2\pi r}\frac{d\Phi}{dt} \tag{8.41}$$

Mathematical Derivation

Betatron condition (2:1 rule):
For stable orbit at constant radius r, we need:
1. Orbital condition:

$$r = \frac{p}{eB_r} \tag{8.42}$$

where B_r is the magnetic field at radius r.
2. Acceleration from induced field:

$$\frac{dp}{dt} = eE = -\frac{e}{2\pi r}\frac{d\Phi}{dt} \tag{8.43}$$

Integrating:

$$p = -\frac{e}{2\pi r}\Phi + \text{const} \tag{8.44}$$

Substituting into orbital condition:

$$r = \frac{1}{eB_r}\left(-\frac{e}{2\pi r}\Phi\right) \tag{8.45}$$

$$2\pi r^2 B_r = \Phi \tag{8.46}$$

Since $\Phi = \pi r^2 \bar{B}$ (average field over area):

$$2\pi r^2 B_r = \pi r^2 \bar{B} \tag{8.47}$$

Important Formula

Betatron Condition:

$$\bar{B} = 2B_r \tag{8.48}$$

The average magnetic field over the orbit area must be exactly twice the field at the orbit radius.

Limitations:

- Energy limited to ∼300 MeV (practical magnet limits)
- Electrons radiate energy (synchrotron radiation) at high energies
- Largely replaced by electron linacs and synchrotrons

Historical significance:

- First successful acceleration to tens of MeV (1940s)
- Important for early nuclear physics research
- Medical applications (X-ray production)

8.5 Particle Detection

Understanding nuclear and particle physics requires not only producing high-energy particles but also detecting and measuring them. Particle detectors exploit the interactions of radiation with matter.

8.5.1 Principles of Charged Particle Detection

When charged particles pass through matter, they interact primarily through:

1. Ionization and Excitation

Charged particles ionize and excite atoms along their path through electromagnetic interactions.

Energy loss (Bethe-Bloch formula):

$$-\frac{dE}{dx} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_ec^2\beta^2\gamma^2T_{max}}{I^2} - \beta^2\right] \tag{8.49}$$

where:

- z = charge of incident particle
- Z, A = atomic number and mass of medium
- $\beta = v/c$
- I = mean excitation energy
- K = constant ≈ 0.307 MeV·cm^2/mol

Key features:

- Energy loss $\propto 1/\beta^2$ at low energies
- Minimum ionization at $\beta\gamma \approx 3 - 4$
- Slow particles ionize more intensely

2. Detection Mechanisms

Detectors exploit different effects of ionization:

1. **Ionization:** Collect ion-electron pairs (gas detectors)
2. **Excitation:** Detect light from de-excitation (scintillators)
3. **Semiconductor excitation:** Create electron-hole pairs (semiconductor detectors)
4. **Chemical changes:** Fog in cloud chamber, grain formation in emulsion
5. **Cherenkov radiation:** Light from particles faster than light in medium

8.5.2 Gas-Filled Detectors

Gas-filled detectors are among the oldest and most widely used radiation detectors.

Basic Operation

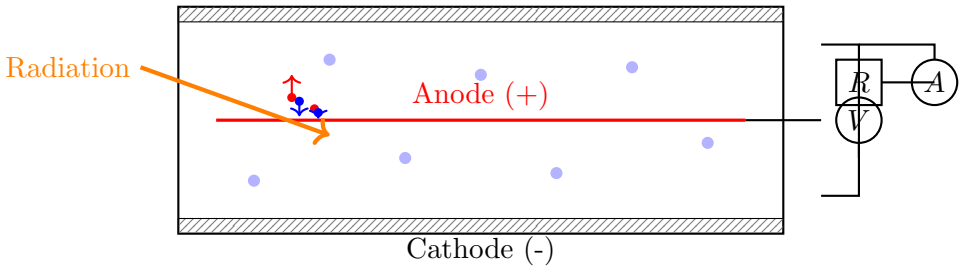


Figure 8.4: Schematic of a gas-filled detector showing ionization by incident radiation, electron drift to anode, and ion drift to cathode.

Process:

1. Radiation ionizes gas atoms: Atom $\rightarrow \text{Ion}^+ + e^-$
2. Electric field separates ions and electrons
3. Electrons drift to anode (fast, $\sim \mu$s)
4. Ions drift to cathode (slow, $\sim$ ms)
5. Current pulse detected

Operating Regions

The behavior depends on applied voltage:

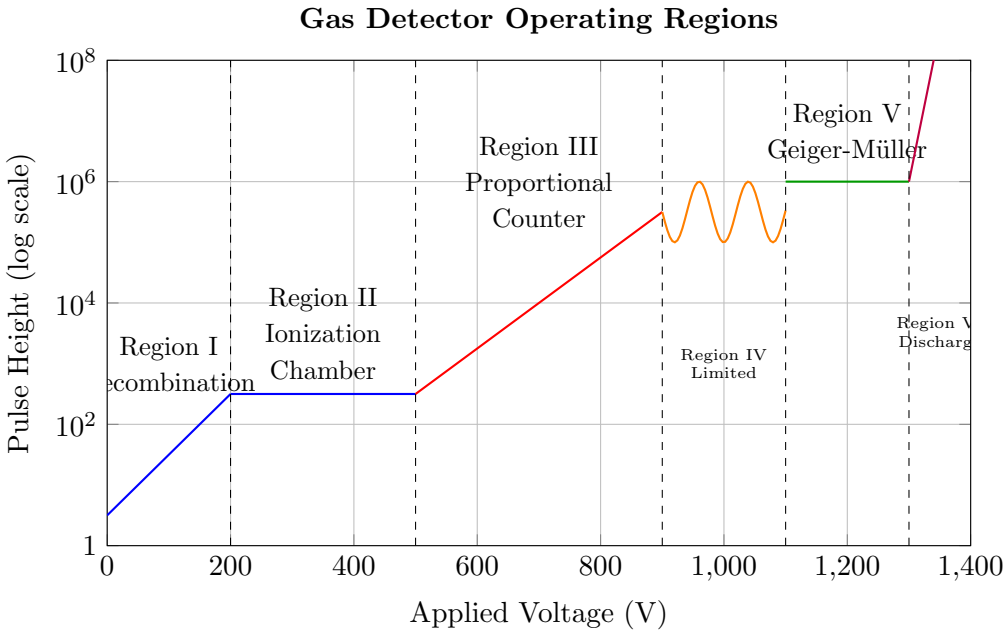


Figure 8.5: Operating regions of gas-filled detectors as a function of applied voltage, showing different detection modes.

8.5.3 Ionization Chamber

Operating region: Voltage sufficient to collect all ions before recombination, but no gas multiplication.

Characteristics:

- Output proportional to ionization (energy deposited)
- No internal amplification
- Requires external amplifier
- Slow response ($\sim$ 1 ms)
- Very stable and reliable

Signal size:
Energy to create one ion pair in air: $W \approx 34$ eV
For 1 MeV particle:

$$N = \frac{E}{W} = \frac{10^6}{34} \approx 29,000 \text{ ion pairs} \tag{8.50}$$

Charge collected:

$$Q = Ne = 29,000 \times 1.6 \times 10^{-19} = 4.6 \times 10^{-15} \text{ C} = 4.6 \text{ fC} \tag{8.51}$$

Very small signal - requires low-noise electronics!

Applications:

- Dose rate meters (radiation protection)
- Smoke detectors (using Am-241)
- Portal monitors

8.5.4 Proportional Counter

Operating region: Higher voltage causes gas multiplication near anode wire, but pulse height still proportional to initial ionization.

Gas Multiplication

Near thin anode wire, electric field is very high:

$$E(r) = \frac{V}{r \ln(b/a)} \tag{8.52}$$

where a = anode radius, b = cathode radius

Avalanche process:

1. Primary electron accelerates in high field
2. Gains enough energy to ionize gas atom
3. Creates secondary electron (and ion)
4. Secondary electron also ionizes
5. Avalanche multiplication occurs

Gas multiplication factor:

$$M = e^{\alpha d} \tag{8.53}$$

where α is Townsend coefficient and d is distance traveled.
Typical values: $M = 10^3$ to 10^6

Key Point

Key Properties of Proportional Counter:

- Output pulse height $\propto$ energy deposited
- Internal amplification ($M \sim 10^3 - 10^6$)
- Can do energy spectroscopy
- Fast response ($\sim 1\mu$s)
- Modest voltage ($\sim$ 1-2 kV)

Applications:

- X-ray spectroscopy
- Neutron detection (with BF_3 or ^{3}He gas)
- Portal monitors
- Medical imaging

8.5.5 Geiger-Müller Counter

The most well-known radiation detector, invented by Hans Geiger and Walther Müller in 1928.

Operating Principle

Operating region: Very high voltage causes complete discharge of counter regardless of initial ionization.

Discharge mechanism:

1. Initial ionization triggers avalanche
2. UV photons from excited gas atoms
3. Photons spread avalanche along entire wire
4. Complete discharge occurs (Geiger discharge)
5. All pulses have same height

Quenching:
Without quenching, continuous discharge would occur. Two methods:
1. Organic quenching gas:

- Add $\sim$10% alcohol, methane, or similar
- Absorbs UV photons
- Prevents photon-induced secondary avalanches

2. Electronic quenching:

- Temporarily reduce voltage after pulse
- Allows ions to clear before next detection

Dead time:
After a pulse, the detector cannot respond for time τ_d (typically 50-200 μs).
True count rate:

$$R_{true} = \frac{R_{observed}}{1 - R_{observed}\tau_d} \tag{8.54}$$

Key Point

Geiger-Müller Counter Characteristics:

- All pulses same height (no energy information)
- Very large signals (easy to detect)
- Simple, rugged design
- Long dead time (limits count rate)
- Wide plateau region (stable operation)
- Typical voltage: 400-1200 V

Applications:

- Portable survey meters
- Contamination monitoring
- Educational demonstrations
- Personal dosimeters

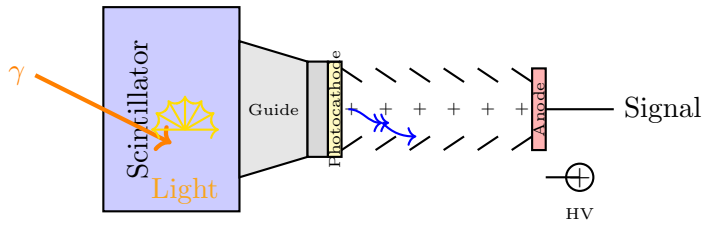
8.5.6 Scintillation Detectors

Scintillation detectors use materials that emit light when radiation passes through them.

Scintillation Mechanism

When radiation deposits energy in a scintillator:

1. Atoms/molecules are excited or ionized
2. De-excitation produces visible or UV photons
3. Photons collected by photomultiplier tube (PMT)
4. PMT converts photons to electrical signal



Scintillation Detector with PMT

Figure 8.6: Schematic of a scintillation detector showing scintillator crystal, light guide, and photomultiplier tube (PMT).

Types of Scintillators

1. Inorganic scintillators:

- **NaI(Tl):** Sodium iodide with thallium activator
  - Light yield: 38,000 photons/MeV
  - Decay time: 230 ns
  - Peak emission: 415 nm
  - Excellent for γ-ray spectroscopy
  - Hygroscopic (must be sealed)
- **CsI(Tl):** Cesium iodide
  - High density (better γ detection)
  - Slightly lower light yield than NaI
  - Less hygroscopic
- **BGO** ($Bi_4Ge_3O_{12}$):
  - Very high density (good stopping power)
  - Lower light yield
  - Used in PET scanners

2. Organic scintillators:

- **Plastic scintillators:**
  - Fast response ($\sim$ 2 ns)
  - Lower light yield
  - Excellent for timing
  - Cheap, can make large sizes
- **Liquid scintillators:**
  - Can dissolve sample directly
  - Used for low-energy β counting
  - Liquid scintillation counting

Photomultiplier Tube (PMT)

Operation:

1. Photons hit photocathode
2. Photoelectric effect releases electrons
3. Electrons accelerated to first dynode
4. Each electron produces ∼3-5 secondary electrons
5. Process repeated through 8-12 dynodes
6. Total gain: 10^6 to 10^8
7. Large pulse collected at anode

Gain:

$$G = \delta^n \tag{8.55}$$

where δ = secondary emission ratio (∼4), n = number of dynodes
For $n = 10$ dynodes:

$$G = 4^{10} \approx 10^6 \tag{8.56}$$

Advantages of scintillation detectors:

- High efficiency for γ-rays (high Z materials)
- Good energy resolution (∼7% for 662 keV with NaI)
- Fast response (ns timing possible)
- Large sizes available
- Can detect neutrons (with ^{6}Li loading)

Applications:

- γ-ray spectroscopy
- Medical imaging (PET, SPECT)
- Well counters
- Portal monitors
- High-energy physics experiments

8.5.7 Semiconductor Detectors

Semiconductor detectors offer the best energy resolution of all detector types.

Basic Principle

In semiconductors, radiation creates electron-hole pairs instead of ion pairs:

- Band gap energy: $E_g \approx 1$ eV (vs. 34 eV in gas)
- More charge carriers per unit energy
- Better energy resolution

Common semiconductors:

- **Silicon (Si):** $E_g = 1.1$ eV, $W = 3.6$ eV/pair
- **Germanium (Ge):** $E_g = 0.7$ eV, $W = 2.9$ eV/pair
- **Cadmium telluride (CdTe):** $E_g = 1.5$ eV, room temperature operation

Detector Structure

Most semiconductor detectors use a **reverse-biased p-n junction**:

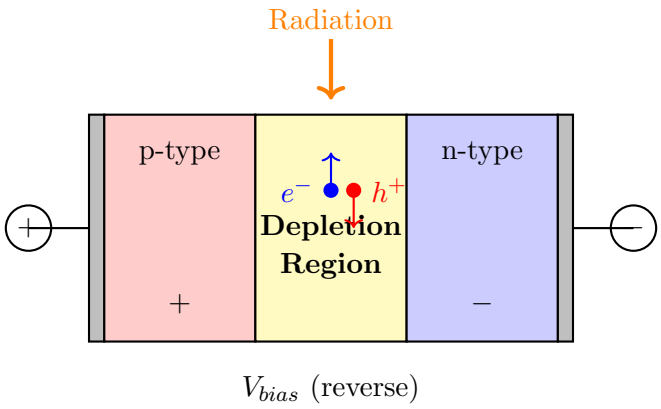


Figure 8.7: Schematic of a semiconductor detector based on reverse-biased p-n junction. Radiation creates electron-hole pairs in depletion region.

Operation:

1. Reverse bias depletes junction of free carriers
2. Radiation creates electron-hole pairs in depletion region
3. Electric field sweeps electrons to n-side, holes to p-side
4. Charge collected produces signal pulse

Depletion depth:

$$d = \sqrt{\frac{2\epsilon V}{eN}} \tag{8.57}$$

where N is the doping concentration.

High-Purity Germanium (HPGe) Detectors

HPGe detectors offer the best energy resolution for γ-ray spectroscopy.

Characteristics:

- Impurity level: $< 10^{10}$ atoms/cm^3
- Large depletion depth (>1 cm possible)
- Must operate at liquid nitrogen temperature (77 K)
- Energy resolution: <2 keV FWHM at 1.33 MeV

Why cooling is needed:

- Small band gap ($E_g = 0.7$ eV)
- Thermal excitation at room temperature creates noise
- Cooling to 77 K reduces leakage current by factor $> 10^6$

Energy resolution comparison:
For 662 keV γ-ray from ^{137}Cs:

- NaI(Tl) scintillator: ∼7% FWHM (∼46 keV)
- HPGe detector: ∼0.3% FWHM (∼2 keV)

Example

Number of charge carriers in semiconductor vs. gas:
For 1 MeV particle:
In gas (air):

$$N_{gas} = \frac{10^6 \text{ eV}}{34 \text{ eV/pair}} = 29,400 \text{ pairs} \tag{8.58}$$

In silicon:

$$N_{Si} = \frac{10^6 \text{ eV}}{3.6 \text{ eV/pair}} = 278,000 \text{ pairs} \tag{8.59}$$

In germanium:

$$N_{Ge} = \frac{10^6 \text{ eV}}{2.9 \text{ eV/pair}} = 345,000 \text{ pairs} \tag{8.60}$$

Statistical fluctuation: $\sigma \propto \sqrt{N}$
Relative resolution: $\frac{\sigma}{N} \propto \frac{1}{\sqrt{N}}$
Ge has ∼ 3.4 times better intrinsic resolution than gas!

Applications of semiconductor detectors:

- High-resolution γ-ray spectroscopy
- X-ray spectroscopy (Si(Li) detectors)
- Charged particle spectroscopy (Si surface barrier)
- Position-sensitive detectors (strip detectors)
- High-energy physics (silicon vertex detectors)

8.6 Comparison of Detector Types

Table 8.2: Comparison of Major Detector Types

Property	Ionization Chamber	Proportional Counter	Scintillator	Semiconductor
Energy per pair	34 eV (air)	30 eV	100 eV (light)	3 eV
Internal gain	1	10^3-10^6	10^6-10^8 (PMT)	1
Energy resolution	Poor	Moderate	Good	Excellent
Response time	Slow (ms)	Fast (μs)	Very fast (ns)	Fast (μs)
Detection efficiency	Low	Low	High (for γ)	Medium
Cost	Low	Low	Moderate	High
Typical use	Dose rate	X-ray spectroscopy	γ spectroscopy	Precision spectroscopy

8.7 Summary

In this chapter, we have explored the essential tools of nuclear physics research: particle accelerators and detectors.

8.7.1 Key Concepts Covered

Particle Accelerators:

- **Linear accelerators:** Accelerate particles in straight line using RF fields; can reach very high energies (e.g., SLAC)
- **Cyclotron:** Uses constant B-field and RF voltage; limited by relativistic effects to $\sim$25 MeV for protons
- **Synchrotron:** Varies B-field to maintain constant orbit radius; achieves highest energies (LHC: 13 TeV)
- **Betatron:** Uses induced electric field from changing flux; limited to $\sim$300 MeV

Particle Detectors:

- **Gas detectors:** Detect ionization in gas; three main types based on voltage:
  - Ionization chamber: No amplification, proportional to energy
  - Proportional counter: Gas multiplication, energy spectroscopy possible
  - Geiger-Müller: Complete discharge, no energy info but very sensitive
- **Scintillation detectors:** Convert radiation to light, amplified by PMT; excellent for γ-rays
- **Semiconductor detectors:** Best energy resolution; HPGe essential for precision spectroscopy

Key Point

Fundamental Relationships:
Accelerators:

- Energy gain: $\Delta E = qV$
- Cyclotron frequency: $f = \frac{qB}{2\pi m}$
- Maximum cyclotron energy: $E_{max} = \frac{q^2 B^2 R^2}{2m}$
- Betatron condition: $\bar{B} = 2B_r$

Detectors:

- Ion pairs in gas: $N = E/W$ where $W \approx 34$ eV
- Semiconductor pairs: $N = E/W$ where $W \approx 3$ eV
- Energy resolution: $\propto 1/\sqrt{N}$

8.8 Practice Problems

1. A cyclotron has a magnetic field of $B = 1.5$ T and a radius of $R = 50$ cm. Calculate:
   (a) The cyclotron frequency for protons
   (b) The maximum kinetic energy of protons
   (c) The number of revolutions needed to reach maximum energy if $V = 50$ kV
2. A 5 MeV alpha particle enters an ionization chamber filled with air at 1 atm. If the chamber volume is 1 liter and all ionization is collected, calculate:
   (a) The number of ion pairs created
   (b) The total charge collected
   (c) The voltage pulse if the chamber capacitance is 100 pF
3. For a proportional counter with gas multiplication factor $M = 10^4$:
   (a) If a 1 MeV gamma ray deposits 100 keV in the counter gas, how many primary ion pairs are created?
   (b) What is the total charge collected after multiplication?
4. A Geiger counter has a dead time of 100 μs. If the observed count rate is 5000 counts per second, what is the true count rate?
5. A NaI(Tl) scintillator produces 38,000 photons per MeV of deposited energy. For a 662 keV gamma ray:
   (a) How many photons are produced?
   (b) If the photomultiplier has 10 dynodes with $\delta = 4$, and photocathode efficiency is 20%, how many electrons arrive at the anode?
6. Calculate the energy resolution (FWHM) for a germanium detector measuring a 1 MeV gamma ray, assuming only statistical fluctuations. Use $W = 2.9$ eV per pair and Fano factor $F = 0.1$.

7. A synchrotron accelerates protons to 7 TeV. Calculate:
   (a) The Lorentz factor γ
   (b) The velocity as a fraction of c
   (c) The momentum
8. Explain why HPGe detectors must be cooled to liquid nitrogen temperature while silicon detectors can operate at room temperature.
9. Compare the number of charge carriers produced by a 1 MeV particle in: (a) air (ionization chamber), (b) silicon detector, (c) germanium detector. How does this affect energy resolution?
10. A betatron operates with maximum magnetic field of 1.0 T at radius $r = 20$ cm. Using the 2:1 rule, calculate:
    (a) The magnetic field at the orbit radius
    (b) The maximum electron energy achievable

End of Chapter 8: Particle Accelerators and Detectors

Chapter 9

Applications of Nuclear Physics

9.1 Introduction

Nuclear physics, once a purely academic pursuit aimed at understanding the fundamental structure of matter, has evolved into a field with profound practical applications that touch nearly every aspect of modern life. From generating electricity to diagnosing and treating diseases, from ensuring food safety to analyzing the age of archaeological artifacts, nuclear techniques have become indispensable tools in science, technology, medicine, and industry.

The applications of nuclear physics can be broadly categorized into:

1. **Energy Production:** Nuclear power plants providing baseload electricity
2. **Medical Applications:** Diagnosis and treatment of diseases
3. **Industrial Applications:** Quality control, materials analysis, and process monitoring
4. **Agricultural Applications:** Crop improvement, pest control, and food preservation
5. **Research Tools:** Advanced analytical techniques like NMR and Mössbauer spectroscopy
6. **Security and Safety:** Radiation protection and nuclear safeguards

In this chapter, we will explore these applications in detail, with particular emphasis on nuclear energy in Pakistan, radiation safety, radioisotope applications, and advanced nuclear techniques.

9.2 Nuclear Energy and Power Generation in Pakistan

9.2.1 Pakistan Atomic Energy Commission (PAEC)

The Pakistan Atomic Energy Commission (PAEC) was established on March 16, 1956, to promote the peaceful uses of nuclear energy and to coordinate nuclear research and development in the country.

Key Point

PAEC Mission:

- Development and promotion of peaceful uses of nuclear energy
- Generation of nuclear electricity
- Nuclear science and technology research
- Medical applications of nuclear technology
- Agricultural and industrial applications
- Human resource development

9.2.2 Nuclear Power in Pakistan

Pakistan has been generating nuclear electricity since 1972. As of 2024, the country operates several nuclear power plants contributing significantly to the national grid.

Operating Nuclear Power Plants

Table 9.1: Nuclear Power Plants in Pakistan (2024)

Plant	Location	Type	Capacity (MW)	Status
KANUPP-1	Karachi	PHWR	137	Decommissioned
CHASNUPP-1	Chashma	PWR	325	Operating
CHASNUPP-2	Chashma	PWR	340	Operating
CHASNUPP-3	Chashma	PWR	340	Operating
CHASNUPP-4	Chashma	PWR	340	Operating
K-2 (KANUPP-2)	Karachi	PWR	1100	Operating
K-3 (KANUPP-3)	Karachi	PWR	1100	Operating
Total Capacity			**3682 MW**	

Reactor Types:

- **PHWR:** Pressurized Heavy Water Reactor (Canadian CANDU design)
- **PWR:** Pressurized Water Reactor (Chinese design - CNP-300 and Hualong One)

Energy Contribution

As of 2024, nuclear power contributes approximately:

- ∼10% of Pakistan's total electricity generation
- ∼3682 MW installed capacity
- Plans to increase to 8800 MW by 2030
- Providing clean, reliable baseload power

9.2.3 Nuclear Fuel Cycle in Pakistan

The nuclear fuel cycle encompasses all activities from uranium mining to waste management.

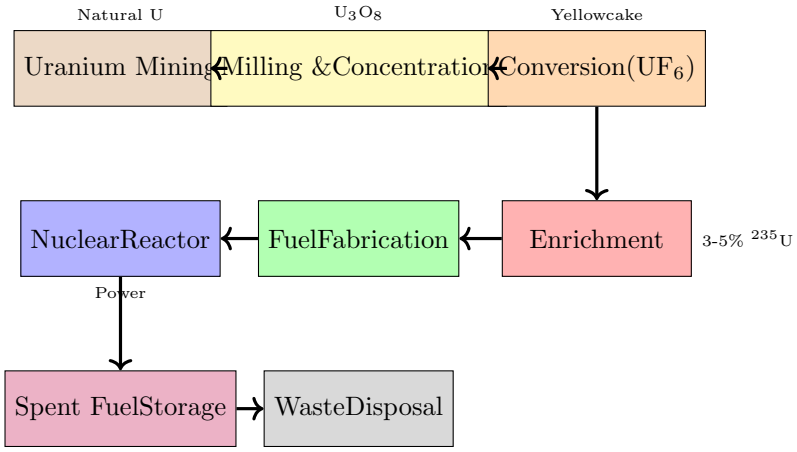


Figure 9.1: Nuclear fuel cycle showing all stages from uranium mining to waste disposal.

PAEC Facilities:

1. **Uranium Mining:** Operations in Punjab and Sindh
2. **Fuel Fabrication:** Kundian Complex (produces fuel assemblies)
3. **Heavy Water Production:** Karachi Nuclear Power Complex
4. **Research Reactors:** PARR-1, PARR-2 (Islamabad)

9.2.4 Safety and Regulation

Pakistan Nuclear Regulatory Authority (PNRA):
Established in 2001, PNRA is the independent regulatory body responsible for:

- Licensing nuclear facilities
- Radiation protection
- Nuclear safety oversight
- Emergency preparedness
- Compliance with IAEA standards

Safety Features of Pakistani Nuclear Plants:

- Multiple redundant safety systems
- Passive cooling systems (K-2, K-3)
- Containment structures
- Emergency core cooling systems
- Regular safety audits and inspections

9.2.5 Energy Economics

Cost Comparison (Approximate):

Table 9.2: Electricity Generation Cost Comparison in Pakistan (PKR/kWh)

Energy Source	Capital Cost	Operating Cost
Nuclear	High	Low (1.0 - 1.5)
Coal	Medium	Medium (6.0 - 8.0)
Gas	Medium	High (10.0 - 15.0)
Hydro	Very High	Very Low (0.5 - 1.0)
Solar	Medium	Low (2.0 - 4.0)
Wind	Medium	Low (3.0 - 5.0)

Key Point

Advantages of Nuclear Power in Pakistan:

- Baseload power generation (90%+ capacity factor)
- Energy security (reduced dependence on imported fuel)
- Low operating costs once plant is built
- Zero greenhouse gas emissions during operation
- Small land footprint compared to renewables
- Job creation and technology development

9.3 Radiation Dosimetry and Protection

9.3.1 Radiation Dose Quantities

Understanding radiation dose is crucial for protecting workers, patients, and the public from harmful radiation exposure.

Absorbed Dose

Definition

Absorbed Dose (D): The energy deposited by radiation per unit mass of material.

$$D = \frac{dE}{dm} \tag{9.1}$$

SI Unit: Gray (Gy) = 1 J/kg
Old Unit: rad (radiation absorbed dose) = 0.01 Gy

Equivalent Dose

Different types of radiation have different biological effectiveness. The equivalent dose accounts for this:

Important Formula

Equivalent Dose:

$$H = D \times w_R \tag{9.2}$$

where w_R is the radiation weighting factor.
SI Unit: Sievert (Sv)
Old Unit: rem (roentgen equivalent man) = 0.01 Sv

Radiation Weighting Factors (w_R):

Table 9.3: Radiation Weighting Factors

Radiation Type	w_R
Photons (X-rays, γ-rays)	1
Electrons, muons	1
Protons	2
Alpha particles	20
Neutrons (energy dependent)	5 - 20
Heavy ions	20

Physical meaning: Alpha particles, despite having short range, cause 20 times more biological damage per unit absorbed dose than gamma rays due to their high ionization density.

Effective Dose

Different organs have different sensitivities to radiation. The effective dose accounts for this:

Important Formula

Effective Dose:

$$E = \sum_T w_T H_T \tag{9.3}$$

where:

- w_T = tissue weighting factor
- H_T = equivalent dose to tissue T

Unit: Sievert (Sv)

Tissue Weighting Factors (w_T):

Table 9.4: Tissue Weighting Factors (ICRP 103)

Tissue/Organ	w_T
Bone marrow, colon, lung, stomach	0.12
Gonads	0.08
Bladder, liver, esophagus, thyroid	0.04
Bone surface, brain, salivary glands	0.01
Skin	0.01
Total	**1.00**

9.3.2 Sources of Radiation Exposure

Average Annual Dose to Public:

Figure 9.2: Sources of radiation exposure to the public (world average ∼5 mSv/year).

Natural Background Radiation:

- **Cosmic rays:** 0.3 - 0.6 mSv/year (higher at altitude)
- **Terrestrial radiation:** 0.3 - 0.6 mSv/year (from rocks and soil)
- **Radon:** 1.0 - 2.0 mSv/year (largest natural source)
- **Internal:** 0.3 - 0.4 mSv/year (K-40, C-14 in body)

9.3.3 Dose Limits and Radiation Protection

ICRP Dose Limits

The International Commission on Radiological Protection (ICRP) recommends dose limits:

Table 9.5: Recommended Dose Limits (ICRP)

Category	Effective Dose (mSv/year)	Equivalent Dose (mSv/year)
Occupational Workers:		
Effective dose	20 (averaged over 5 years)	-
Eye lens	-	20
Skin	-	500
Hands and feet	-	500
General Public:		
Effective dose	1	-
Eye lens	-	15
Skin	-	50

ALARA Principle

Key Point

ALARA: As Low As Reasonably Achievable
All radiation exposures should be kept as low as reasonably achievable, taking into account:

- Economic factors
- Technological factors
- Social factors
- Benefits vs. risks

Three Principles of Radiation Protection

1. Time:
Minimize exposure time. Dose is proportional to exposure time:

$$D \propto t \tag{9.4}$$

2. Distance:
Maximize distance from source. For point sources, dose follows inverse square law:

$$D \propto \frac{1}{r^2} \tag{9.5}$$

Example

Inverse Square Law Application:
If dose rate at 1 meter from a source is 100 mSv/h, what is it at 4 meters?
Solution:

$$\frac{D_2}{D_1} = \frac{r_1^2}{r_2^2} \tag{9.6}$$

$$D_2 = D_1 \times \frac{r_1^2}{r_2^2} = 100 \times \frac{1^2}{4^2} \tag{9.7}$$

$$= 100 \times \frac{1}{16} = 6.25 \text{ mSv/h} \tag{9.8}$$

Increasing distance by factor of 4 reduces dose rate by factor of 16.

3. Shielding:
Use appropriate shielding materials:

Table 9.6: Shielding Materials for Different Radiations

Radiation	Best Shielding	Typical Thickness
Alpha (α)	Paper, skin	< 0.1 mm
Beta (β)	Plastic, aluminum	Few mm
Gamma (γ)	Lead, concrete	cm to meters
Neutrons	Water, concrete, polyethylene	Tens of cm

Attenuation of gamma rays:

$$I(x) = I_0 e^{-\mu x} \tag{9.9}$$

where μ is the linear attenuation coefficient.
Half-value layer (HVL):

$$\text{HVL} = \frac{\ln 2}{\mu} = \frac{0.693}{\mu} \tag{9.10}$$

9.3.4 Radiation Detection and Monitoring

Personal Dosimetry Devices:

1. **Film Badge:** Photographic film darkens with exposure
2. **Thermoluminescent Dosimeter (TLD):** Crystal stores energy, releases light when heated

3. **Electronic Personal Dosimeter (EPD):** Real-time digital readout
4. **Pocket Ionization Chamber:** Direct reading dosimeter

Area Monitoring:

- Survey meters (Geiger-Müller, scintillation)
- Continuous air monitors
- Contamination monitors
- Portal monitors

9.4 Radioisotopes in Medicine, Agriculture, and Industry

9.4.1 Medical Applications

Nuclear medicine uses radioisotopes for both diagnosis and treatment of diseases.

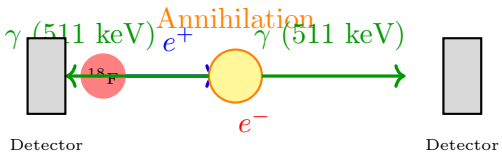
Diagnostic Applications

Nuclear Imaging Techniques:

1. Gamma Camera (Scintigraphy):
Uses gamma-emitting radiopharmaceuticals to create images of organs and tissues.
Common radioisotopes:

- ^{99m}**Tc** (Technetium-99m): Most widely used
  - $t_{1/2} = 6.02$ hours
  - Gamma energy: 140 keV (ideal for imaging)
  - Used in 80% of nuclear medicine procedures
  - Bone scans, cardiac imaging, kidney function
- ^{131}**I** (Iodine-131):
  - $t_{1/2} = 8.02$ days
  - Thyroid imaging and treatment
- ^{201}**Tl** (Thallium-201):
  - $t_{1/2} = 73$ hours
  - Cardiac perfusion imaging

2. Positron Emission Tomography (PET):
Uses positron-emitting radioisotopes. When positron annihilates with electron, two 511 keV gamma rays are emitted in opposite directions.



PET Imaging Principle

Figure 9.3: Principle of PET imaging: positron-electron annihilation produces two 511 keV photons traveling in opposite directions.

Common PET isotopes:

- ^{18}**F**: $t_{1/2} = 110$ min (FDG - cancer detection)
- ^{11}**C**: $t_{1/2} = 20$ min
- ^{13}**N**: $t_{1/2} = 10$ min
- ^{15}**O**: $t_{1/2} = 2$ min

Therapeutic Applications

1. Targeted Radionuclide Therapy:

- ^{131}**I**: Thyroid cancer and hyperthyroidism
  - Beta emissions kill thyroid cells
  - Typical dose: 1-6 GBq
- ^{177}**Lu**: Neuroendocrine tumors
  - $t_{1/2} = 6.65$ days
  - Beta range: ~ 2 mm (localized damage)
- ^{90}**Y**: Liver cancer, lymphoma
  - $t_{1/2} = 64$ hours
  - Pure beta emitter

2. Brachytherapy:
Sealed radioactive sources placed inside or near tumor:

- ^{192}**Ir**: Prostate, breast cancer
- ^{125}**I**: Prostate cancer seeds
- ^{60}**Co**: External beam therapy

3. Radiation Dose Calculation in Therapy:

Mathematical Derivation

Cumulated Activity:
The total number of decays in tissue:

$$\tilde{A} = \int_0^\infty A(t)dt \tag{9.11}$$

For simple exponential decay:

$$\tilde{A} = A_0 \int_0^\infty e^{-\lambda t}dt = \frac{A_0}{\lambda} = A_0\tau \tag{9.12}$$

where $\tau = 1/\lambda$ is the mean life.
Absorbed Dose:

$$D = \frac{\tilde{A} \times E \times \phi}{m} \tag{9.13}$$

where:

- E = energy per decay
- ϕ = fraction of energy absorbed
- m = mass of target tissue

9.4.2 Agricultural Applications

Crop Improvement

1. Mutation Breeding:
Irradiation induces genetic mutations to develop new crop varieties:

- Gamma rays from ^{60}Co or ^{137}Cs
- Typical doses: 100-500 Gy for seeds
- Over 3,000 mutant varieties developed worldwide
- Examples: disease-resistant rice, high-yield wheat

2. Tracer Studies:
Radioactive tracers study:

- Fertilizer uptake (using ^{32}P, ^{15}N)
- Water movement in plants
- Pesticide distribution
- Root development

Pest Control

Sterile Insect Technique (SIT):

Practical Application

Sterile Insect Technique Process:

1. Mass-rear target insect species
2. Irradiate pupae with gamma rays (100-150 Gy)
3. Sterilize males without affecting mating competitiveness
4. Release sterile males in large numbers
5. Sterile males mate with wild females
6. No offspring produced
7. Pest population declines

Successes:

- Eradication of screwworm in North America
- Control of Mediterranean fruit fly
- Tsetse fly control in Africa

Food Preservation

Food Irradiation:
Uses ionizing radiation to:

- Kill bacteria, parasites, and insects
- Delay ripening and sprouting
- Extend shelf life
- Reduce foodborne illness

Typical doses:

- **Low dose (¡ 1 kGy):** Delay ripening, inhibit sprouting
- **Medium dose (1-10 kGy):** Reduce pathogens, extend shelf life
- **High dose (¿ 10 kGy):** Sterilization for astronaut food

Advantages:

- No chemical residues
- Treats packaged food
- More effective than heat for some foods
- Environmentally friendly

Safety: Food does not become radioactive. Approved by WHO, FDA, and over 60 countries.

9.4.3 Industrial Applications

Radiography

Industrial Radiography:

X-rays and gamma rays inspect:

- Welds in pipelines and pressure vessels
- Aircraft components
- Structural integrity
- Castings for defects

Common sources:

- ^{192}**Ir**: $t_{1/2} = 74$ days, 0.31-0.61 MeV γ
- ^{60}**Co**: $t_{1/2} = 5.27$ years, 1.17-1.33 MeV γ
- ^{137}**Cs**: $t_{1/2} = 30$ years, 0.66 MeV γ

Gauging

1. Thickness Gauging:

Beta or gamma radiation measures material thickness:

$$I = I_0 e^{-\mu \rho x} \tag{9.14}$$

where:

- I = transmitted intensity
- μ = mass attenuation coefficient
- ρ = density
- x = thickness

Applications:

- Paper thickness in mills
- Steel sheet thickness
- Plastic film production
- Coating thickness

2. Level Gauging:

Measures liquid or solid levels in tanks and silos.

3. Density Gauging:

Measures soil compaction, concrete density, etc.

Neutron Activation Analysis (NAA)

Principle:

Sample irradiated with neutrons becomes radioactive:

$$n + {}^{A}X \rightarrow {}^{A+1}X^{*} \rightarrow {}^{A+1}X + \gamma \tag{9.15}$$

Process:

1. Irradiate sample in neutron source (reactor)
2. Induced radioactivity measured with gamma spectrometer
3. Identify elements from gamma ray energies
4. Quantify from activity levels

Advantages:

- Non-destructive
- High sensitivity (ppb to ppt)
- Multi-element analysis
- Minimal sample preparation

Applications:

- Archaeology (dating, provenance)
- Forensics (gunshot residue, trace evidence)
- Environmental monitoring
- Quality control

9.5 Nuclear Magnetic Resonance (NMR)

9.5.1 Principles of NMR

Nuclear Magnetic Resonance is a quantum mechanical phenomenon where certain nuclei absorb and re-emit electromagnetic radiation in a magnetic field.

Nuclear Spin and Magnetic Moment

Nuclei with odd mass number or odd atomic number possess nuclear spin I:

Key Point

Nuclear Spin:

- Nuclei with even A and even Z: $I = 0$ (no NMR signal)
- Nuclei with odd A or odd Z: $I \neq 0$ (NMR active)
- Examples: ^{1}H ($I = 1/2$), ^{13}C ($I = 1/2$), ^{14}N ($I = 1$)

The nuclear magnetic moment:

$$\vec{\mu} = \gamma \vec{I} \tag{9.16}$$

where γ is the gyromagnetic ratio (nucleus-specific).

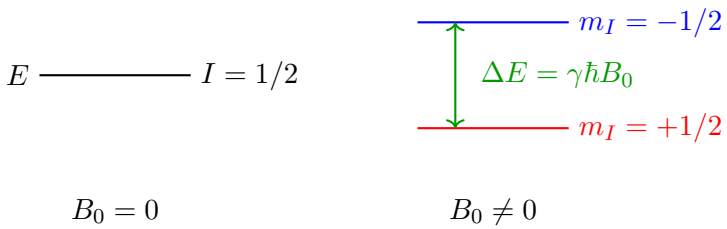
Zeeman Splitting

In an external magnetic field B_0, the energy levels split:

$$E_m = -\gamma \hbar B_0 m_I \tag{9.17}$$

where $m_I = -I, -I+1, \ldots, I-1, I$
For spin-1/2 nuclei ($m_I = \pm 1/2$):

$$\Delta E = \gamma \hbar B_0 \tag{9.18}$$



Zeeman Splitting in Magnetic Field

Figure 9.4: Energy level splitting for spin-1/2 nucleus in magnetic field.

Resonance Condition

Resonance occurs when the applied RF frequency matches the Larmor frequency:

Important Formula

Larmor Frequency:

$$\nu_0 = \frac{\gamma B_0}{2\pi} \tag{9.19}$$

or in angular frequency:

$$\omega_0 = \gamma B_0 \tag{9.20}$$

Example

Calculate the NMR frequency for protons in a 1.5 T magnetic field:
For ^{1}H: $\gamma = 2.675 \times 10^8$ rad/(s·T)
Solution:

$$\nu_0 = \frac{\gamma B_0}{2\pi} \tag{9.21}$$

$$= \frac{2.675 \times 10^8 \times 1.5}{2\pi} \tag{9.22}$$

$$= 6.38 \times 10^7 \text{ Hz} = 63.8 \text{ MHz} \tag{9.23}$$

This is in the radio frequency range.

9.5.2 NMR Spectroscopy

Chemical Shift

Electrons around nucleus create local magnetic fields, slightly altering the effective field:

$$B_{\text{eff}} = B_0(1 - \sigma) \tag{9.24}$$

where σ is the shielding constant.

Chemical shift:

$$\delta = \frac{\nu_{\text{sample}} - \nu_{\text{ref}}}{\nu_{\text{ref}}} \times 10^6 \text{ (ppm)} \tag{9.25}$$

Different chemical environments give different chemical shifts, allowing molecular structure determination.

Spin-Spin Coupling

Nuclear spins interact through bonding electrons, causing peak splitting:

$n + 1$ **rule:** A nucleus with n equivalent neighbors shows $n + 1$ peaks.

9.5.3 Magnetic Resonance Imaging (MRI)

MRI uses NMR principles to create detailed images of internal body structures.

Basic Components:

1. **Main magnet:** 0.5 - 7 T (typically 1.5 - 3 T)
2. **Gradient coils:** Create spatial variation in field
3. **RF coils:** Transmit and receive RF signals
4. **Computer:** Image reconstruction

Advantages over X-ray CT:

- No ionizing radiation
- Better soft tissue contrast
- Any plane imaging
- Functional imaging possible (fMRI)

Contrast mechanisms:

- **T1 relaxation:** Spin-lattice relaxation
- **T2 relaxation:** Spin-spin relaxation
- **Proton density:** Number of hydrogen atoms

9.6 Mössbauer Effect

9.6.1 Discovery and Principle

The Mössbauer effect, discovered by Rudolf Mössbauer in 1958 (Nobel Prize 1961), is the recoil-free emission and absorption of gamma rays by nuclei bound in a solid lattice.

Recoil Problem in Free Atoms

When a free atom emits a gamma ray:

Mathematical Derivation

Energy and momentum conservation:

$$E_\gamma + E_R = E_0 \tag{9.26}$$

$$p_\gamma = p_{\text{recoil}} \tag{9.27}$$

The recoil energy:

$$E_R = \frac{p^2}{2M} = \frac{E_\gamma^2}{2Mc^2} \tag{9.28}$$

For ^{57}Fe emitting 14.4 keV gamma ray ($M = 57$ u):

$$E_R = \frac{(14.4 \times 10^3)^2}{2 \times 57 \times 931.5 \times 10^6} = 1.95 \times 10^{-3} \text{ eV} \tag{9.29}$$

This energy is lost from the gamma ray, and natural linewidth is $\sim 10^{-8}$ eV.
Without recoil compensation, emission and absorption lines don't overlap!

Mössbauer Effect Solution

In a solid lattice at low temperature:

- Recoil momentum absorbed by entire crystal
- Effective mass $M \to M_{\text{crystal}} \approx 10^{23}$ u
- Recoil energy becomes negligible
- Some fraction of emissions are recoil-free

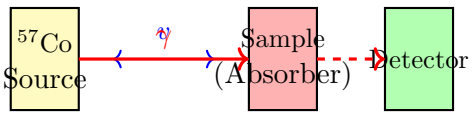
Recoil-free fraction (Lamb-Mössbauer factor):

$$f = e^{-\frac{E_R^2}{\hbar^2 \omega_D^2}} \tag{9.30}$$

where ω_D is the Debye frequency.

9.6.2 Mössbauer Spectroscopy

Experimental Setup:



Mössbauer Spectrometer

Figure 9.5: Basic Mössbauer spectroscopy setup. Source moves with velocity v to vary gamma ray energy via Doppler effect.

Doppler Velocity Modulation:

Moving source with velocity v shifts gamma ray energy:

$$E_\gamma = E_0 \left(1 + \frac{v}{c}\right) \tag{9.31}$$

Typical velocities: ± few mm/s

Hyperfine Interactions

Mössbauer spectroscopy reveals:

1. Isomer Shift (δ):
Shift due to different electron density at nucleus in source vs. absorber.

2. Quadrupole Splitting (ΔE_Q):
Interaction of nuclear quadrupole moment with electric field gradient:

$$\Delta E_Q = \frac{1}{2} eQV_{zz} \left(1 + \frac{\eta^2}{3}\right)^{1/2} \tag{9.32}$$

Reveals asymmetry of charge distribution.

3. Magnetic Hyperfine Splitting:
Interaction with internal magnetic field splits levels into $2I + 1$ sublevels.
For ^{57}Fe ($I = 1/2$ ground state, $I = 3/2$ excited state):

- Ground state splits into 2 levels
- Excited state splits into 4 levels
- 6 allowed transitions
- Sextet pattern observed

9.6.3 Applications of Mössbauer Spectroscopy

1. Chemistry:

- Oxidation state determination
- Chemical bonding studies
- Coordination chemistry

2. Physics:

- Magnetic properties of materials
- Superconductivity studies
- Lattice dynamics

3. Geology:

- Mineral identification
- Meteorite analysis
- Lunar samples

4. Biology:

- Metalloprotein structure
- Iron in biological systems
- Enzyme studies

9.7 Summary

This chapter has explored the diverse applications of nuclear physics:

1. **Nuclear Energy in Pakistan:**
   - PAEC operates 3682 MW of nuclear capacity
   - PWR and PHWR reactor technologies
   - Plans to expand to 8800 MW by 2030
   - Provides clean, reliable baseload power
2. **Radiation Dosimetry and Protection:**
   - Absorbed dose (Gy), equivalent dose (Sv), effective dose (Sv)
   - ICRP dose limits: 20 mSv/year for workers, 1 mSv/year for public
   - ALARA principle and three protection methods (time, distance, shielding)
   - Personal and area monitoring devices
3. **Radioisotope Applications:**
   - **Medicine:** Diagnosis (Tc-99m, PET) and therapy (I-131, Lu-177)
   - **Agriculture:** Mutation breeding, SIT pest control, food irradiation
   - **Industry:** Radiography, gauging, neutron activation analysis
4. **Advanced Techniques:**
   - **NMR:** Based on nuclear spin resonance in magnetic field
   - MRI provides detailed medical imaging without ionizing radiation
   - **Mössbauer Effect:** Recoil-free gamma emission/absorption
   - Reveals hyperfine interactions and material properties

Key Point

Nuclear physics applications have transformed modern society:

- Clean energy production
- Advanced medical diagnosis and treatment
- Food safety and preservation
- Materials analysis and quality control
- Fundamental research tools

As technology advances, new applications continue to emerge, demonstrating the enduring importance of nuclear physics in addressing global challenges.

9.8 Problems

1. If Pakistan's nuclear power plants operate at 90% capacity factor, how much electricity do they generate annually? Compare this to the energy from burning coal (assume coal plants at 40% efficiency).

2. A radiation worker receives an equivalent dose of 2 mSv from gamma rays and 0.5 mSv from neutrons (assume $w_R = 10$ for neutrons). Calculate:

   (a) The absorbed dose from each source

   (b) The total effective dose

   (c) Is this within annual limits?

3. The dose rate at 2 m from a gamma source is 10 mSv/h. Calculate:

   (a) Dose rate at 5 m

   (b) Safe working time at 2 m to receive < 0.1 mSv

   (c) Minimum safe distance for 1-hour exposure

4. A 70 kg patient receives 500 MBq of Tc-99m for a bone scan. If 30% localizes in bone and the effective half-life in bone is 5 hours, calculate the cumulated activity in bone.

5. Calculate the thickness of lead required to reduce 1 MeV gamma ray intensity by 90%. (Linear attenuation coefficient for Pb at 1 MeV: $\mu = 0.77$ cm^{-1})

6. For proton NMR at 7 T (ultra-high field MRI), calculate:

   (a) The Larmor frequency

   (b) The energy difference between spin states

   (c) Compare to thermal energy at body temperature (37°C)

7. Explain why the Mössbauer effect is not observed in gases but is prominent in solids at low temperature.

8. If 1000 kg of potatoes are irradiated with 1 kGy dose to inhibit sprouting, and the irradiation source delivers 5 kGy/hour, calculate:

   (a) Energy deposited in potatoes

   (b) Time required for treatment

   (c) Compare energy to cooking the potatoes (assume 2 kJ/kg to heat by 1°C)

9. In neutron activation analysis, a 1 mg sample of gold is irradiated in a neutron flux of 10^{13} n/(cm^2·s) for 1 hour. Given: $\sigma = 98.8$ barn for $^{197}\text{Au}(n,\gamma)^{198}\text{Au}$, and $t_{1/2} = 2.7$ days for ^{198}Au, calculate the induced activity.

10. Discuss the relative advantages and disadvantages of nuclear power versus renewable energy (solar, wind) for Pakistan's energy security.

End of Chapter 9: Applications of Nuclear Physics

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